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EXPERIMENTAL ANALYSIS OF A NEW APPROXIMATION TO THE SURGE DISTR--ETC(U)
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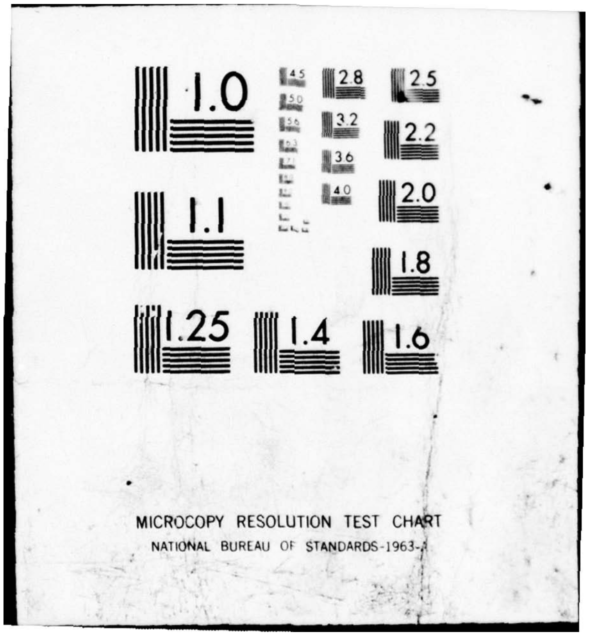
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EXPERIMENTAL ANALYSIS OF A NEW APPROXIMATION TO THE SURGE DISTRIBUTION.

by

10 Vernon Roy Brethour

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Experimental Analysis of a New Approximation to the Surge Distribution

Thesis directed by Professor Petr Beckmann

A new simplified approximation to the surge distribution of a random process has been proposed. In this paper, the approximation is experimentally checked by computer simulation of both normal and exponential processes. It is found that the accuracy of the approximation depends on the autocorrelation function of the process.

This abstract is approved as to form and content.

Signed Petr Beckmann
Faculty member in charge of thesis

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CHAPTER I

INTRODUCTION

In applications of the theory of random processes, a function which is often important is the probability density function of the duration τ, of surges of a random process above a given critical value. A practical method for deriving this probability density $p_\tau(\alpha)$, for a given random process remains unknown. Thus the engineer who needs to know $p_\tau(\alpha)$ must be content, instead, with approximations to it.[1] The approximation most widely accepted was published in 1958 by S. O. Rice.[2] This approximation has been proved to be accurate, but it is difficult to obtain. In addition, Rice's work only applies directly to Gaussian processes and is different to extend to non-Gaussian processes.

A. N. Denisenko has recently proposed a new approximation which is far easier to evaluate for Gaussian processes and which can be

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[1]The notation $p(\tau)$ is often used for this distribution, but to avoid confusion between the random variable and the argument of its distribution function, $p_\tau(\alpha)$ will be used in this paper. With this notation, the probability of τ being less than x is

$$\int_{-\infty}^{x} P_\tau(\alpha)d\alpha$$

[2]Stephan O. Rice. "Distribution of the Duration of Fades in Radio Transmission: Gaussian Noise Model." <u>The Bell System Technical Journal</u>. Vol. 37, No. 3 (May 1958), pp. 581-635.

easily extended to non-Gaussian processes.[3] In his presentation of this new approximation, Denisenko included computer simulation data which tended to verify the accuracy of his approximation.

The purpose of this thesis is to check the results of Denisenko's approximation against experimental data, including non-Gaussian processes.

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[3]A. N. Denisenko. "Estimate of the Distribution of Surge Durations for Random Processes." Radiotekhn, i Elektr. Vol. 20, (July 1975), pp. 1529-1532.

CHAPTER II

THEORETICAL BACKGROUND

The Mean of the Surge Distribution

While the exact solution for $p_\tau(\alpha)$ remains unknown, the mean of the distribution is known. For a random process $X(t)$, which at any given time has probability distribution $p_x(\beta)$ the mean duration of a surge above level x_o is

$$E[\tau] = \frac{P(X>x_o)}{\frac{1}{2}\nu(x_o)} \tag{1}$$

where

$$P(X>x_o) = \int_{x_o}^{\infty} p_x(\beta)\, d\beta$$

and ν is the average frequency with which X crosses x_o given by

$$\nu(x_o) = \int_{-\infty}^{\infty} \gamma p_{x,x'}(x_o,\gamma)\, d\gamma$$

where $p_{x,x'}(a,b)$ is the two dimensional probability distribution of $X(t)$ and $X'(t)$.[1]

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[1]Petr Beckmann. Probability in Communication Engineering. New York, Harcourt, Brace & World Inc., 1967, p. 229. E[] is the expected value operator, and a prime indicates the derivative with respect to the argument.

For a normal process having mean μ, variance σ^2, correlation function $B(\tau) = E[X(t)\ X(t+\tau)]$, and correlation coefficient $R(\tau) = B(\tau)/\sigma^2$; equation (1) gives[2]

$$E[\tau] = \frac{\pi}{\sqrt{-R''(0)}} \exp \frac{(x_o-\mu)^2}{2\sigma^2} \operatorname{erfc} \frac{x_o-\mu}{\sigma\sqrt{2}} \tag{2}$$

where

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-\alpha^2} d\alpha$$

If the process $X(t)$ is stationary, exponential, "analogus to a normal process", and with derivative $X'(t)$ independent of $X(t)$, then $R(\tau) = \nu^2 B(\tau)$ and[3]

$$E[\tau] = \sqrt{\frac{8}{-R''(0)}} \tag{3}$$

It is interesting to note that this mean value is independent of the surge level x_o.

While these results are important and useful, $p_\tau(\alpha)$ must still be estimated because the above calculations yield only the mean of the distribution and say nothing about how $p_\tau(\alpha)$ is distributed about the mean.

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[2]Ibid. p. 230.

[3]Petr Beckmann. Orthogonal Polynomials for Engineers and Physicists. Boulder, Colorado, The Golem Press, 1973, p. 196. The expression, "analogus to a normal process", means that the two dimensional density of the process can be expanded in orthogonal polynomials in a cononical form as explained in this reference.

Rice's Approximation

The approximation to the surge distribution which Rice proposed is actually the combination of two approximations. First Rice developed a probability function $p_\tau^{(1)}(\alpha)$ which holds exactly only for infinitesmally small values of α. This function then becomes the approximation to $p_\tau(\alpha)$ for all "small to medium" values of α, including those significantly greater than zero. The expression for $p_\tau^{(1)}(\alpha)$ is[4]

$$p_\tau^{(1)}(\alpha) = M_{22}\Omega^{-1/2}(1-B^2(\alpha))^{-3/2}\exp[\frac{x_o^2}{2} - \frac{x_o^2}{1+B(\alpha)}]J(r,h) \qquad (4)$$

where

$$M_{22} = (1-B^2(\alpha)) - B'(\alpha)$$

$$\Omega = \sqrt{-R''(0)}$$

$$h = \frac{x_o B'(\alpha)}{1+B(\alpha)}\left[\frac{1-B^2(\alpha)}{M_{22}}\right]^{1/2}$$

$$r = \frac{B''(\tau)(1-B^2(\alpha)) + B(\alpha)[B'(\alpha)]^2}{(1-B^2(\alpha)) - [B'(\alpha)]^2}$$

$$J(r,h) = \frac{1}{2\pi s}\int_h^\infty\int_h^\infty (x-h)(y-h)e^{-z}dy\,dx$$

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[4]Stephan O. Rice. "Distribution of the Duration of Fades in Radio Transmission: Gaussian Noise Model." The Bell System Technical Journal, Vol. XXXVIII, No. 3 (May 1958), p. 601.

$$s = (1-r^2)^{1/2}$$

$$z = \frac{x^2+y^2-2rxy}{2(1-r^2)}$$

Next Rice approximates the distribution of very long surges (large values of α where $p_\tau^{(1)}(\alpha)$ is no longer even closed to valid) by

$$p_\tau^{(2)}(\alpha) = K_1 e^{-K_2\alpha} \tag{5}$$

These two approximations (equations (4) and (5)) are then plotted and a value of α is picked below which (4) will apply and above which (5) will hold. At the same time K_1 and K_2 are adjusted so that (i) the resulting curve looks "reasonable", (ii) the resulting approximation $\hat{p}_\tau(\alpha)$ has the proper mean (given by equation (2)), and finally (iii) $\int_0^\infty \hat{p}_\tau(\alpha)d\alpha = 1$. All of this is done by eye, so we see that artwork and experience as well as mathematics play a role in obtaining $\hat{p}_\tau(\alpha)$ by Rice's method. It has been amply verified that approximations obtained by Rice's method agree well with experimental data.[5]

Denisenko's Approximation

Denisenko approached the problem by estimating the cumulative distribution function $F_\tau(\alpha) = \int_0^\alpha p_\tau(\gamma)d\gamma$ which is equal to the

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[5]Ibid. pp. 589-599.

probability of a surge of length $\tau \leq \alpha$, which is equal to 1 minus the probability of a surge of length $\tau > \alpha$. The exact expression for the probability of a surge of length $\tau > \alpha$ can be expressed as[6]

$$\lim_{n\to\infty}[P(X(t_2)>x_o,\ X(t_3)>x_o, \ldots X(t_n)>x_o | X(t_1)>x_o)] \tag{6}$$

where $t_n - t_1 = \alpha$ and the t_k are all chosen so that $t_{k+1} - t_k = \alpha/n$. Let U_k be the event $X(t_k)>x_o$. Using this to rewrite equation (6) gives the cumulative distribution

$$F_\tau(\alpha) = 1 - \lim_{n\to\infty}[P(U_2|U_1)\cdot P(U_3|U_1,U_2)\ldots P(U_n|U_1,U_2,U_3,\ldots U_{n-1})] \tag{7}$$

At this point, Denisenko states without proof that

$$P(U_k|U_1,U_2,U_3,\ldots U_{k-1}) \leq P(U_k|U_{k-1}) \tag{8}$$

He then uses the lower bound of this expression as an approximation to the left side, ie.

$$P(U_k|U_1,U_2,U_3,\ldots U_{k-1}) \simeq P(U_k|U_{k-1}) \tag{9}$$

Equation (9) applied n-2 times in equation (7) gives

$$\hat{F}_\tau(\alpha) = 1 - \lim_{n\to\infty}[P(U_n|U_{n-1})]^n$$

or

$$\hat{F}_\tau(\alpha) = 1 - \lim_{n\to\infty}[P(X(t_k) > x_o|X(t_{k-1}) > x_o)]^n$$

---

[6] P( ) is the probability of the argument in brackets, a vertical bar means "given that", and a comma means "and".

or

$$\hat{F}_\tau(\alpha) = 1 - \lim_{n\to\infty}[P(X(t+\frac{\alpha}{n}) > x_o | X(t) > x_o)]^n$$

and finally, Bayes Rule yields

$$\hat{F}_\tau(\alpha) = 1 - \lim_{n\to\infty}[\frac{P(X(t+\frac{\alpha}{n}) > x_o, X(t) > x_o)}{P(X(t) > x_o)}]^n \tag{10}$$

Up to this point, the density $p_x(\alpha)$ of the process X(t) has not been considered. Now equation (10), when applied to a normal process with mean μ and variance σ^2 gives (following a Taylor series expansion about R(0) and some algebra)

$$\hat{F}_\tau(\alpha) = 1-e^{-A\alpha} \tag{11}$$

where

$$A(x_o) = \frac{\sqrt{-R''(0)}}{2\pi[1-\Phi\frac{(x_o-\mu)}{\sigma}]} \exp[-\frac{(x_o-\mu)^2}{2\sigma^2}] \tag{12}$$

and

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt \tag{13}$$

And finally taking the derivative of equation (11) with respect to α, yields the estimate of the density of τ.

$$\hat{p}_\tau(\alpha) = Ae^{-A\alpha} \tag{14}$$

For an unbiased approximation, the mean of this estimate should agree with the known mean of the surge length distribution. As shown in Appendix E, this is the case.

The Extension of Denisenko's Approximation

The direct application of equation (10) to non-normal processes results in expressions which cannot be evaluated in closed form. However, working directly from equation (12) Beckmann extended Denisenko's results to non-normal processes.[7]

Suppose that the surge distribution of a process $Y(t)$ is required, where $Y(t)$ has arbitrary probability distribution $p_Y(\alpha)$ and corresponding cumulative distribution function $F_Y(\alpha)$ $= \int_{-\infty}^{\gamma} p_Y(\gamma)d\gamma$. Let the process $X(t)$ be normally distributed with zero mean and unit variance. Thus $X(t)$ has cumulative distribution function $\Phi(\alpha)$ given by equation (13).

The process $Y(t)$ is obtained as some monotonic function $G(X)$ of $X(t)$, and $G(X)$ is picked so that $Y(t)$ has the desired distribution function.

$$Y(t) = G(X(t))$$

so

$$P(X \leq \alpha) = P(Y \leq G(\alpha))$$

$$\Phi(\alpha) = F_Y(G(\alpha))$$

or

$$G(\alpha) = F_Y^{-1}(\Phi(\alpha)) \tag{15}$$

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[7]Petr Beckmann. "Probability Distribution of Surges and Fades." Proceedings of the IEEE, Vol. 64, No. 4, (April 1976), pp. 571-572.

In this way it is always possible to find a function G which transforms a normal distribution to the desired distribution of Y. The duration of a surge of X(t) above level x_o will be equal to the duration of a surge of y(t) above level $G(x_o)$. Thus to get the surge distribution of Y(t) above level y_o, equation (11) can be used with the x_o terms in equation (12) replaced by $G^{-1}(y_o)$. It is important to remember that the term $\sqrt{-R''(0)}$ in equation (12) now refers to the process Y(t) and $R_Y(\tau)$ must be evaluated from

$$R_y(\tau) = \frac{1}{\sigma_y^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{G(x_1)G(x_2)}{2\pi\sqrt{1-R_x^2(\tau)}}\right] \cdot$$

$$\exp\left[\frac{x_1^2-2R_x(\tau)x_1x_2-x_2^2}{2(1-R_x^2(\tau))}\right]dx_1dx_2 \tag{16}$$

where σ_y is the variance of Y(t).

The evaluation of equation (16) can be simplified somewhat by use of orthogonal polynomials. However, even if orthogonal polynomials are used, evaluating this equation (since the second derivative of the result is required), and finding $G(\alpha)$ can involve difficult computations. Fortunately a simpler method exists which can eliminate these cumbersome calculations.

The mean of the surge distribution of Y(t) can be calculated directly from a knowledge of Y(t) using equation (1). Because the mean of Denisenko's approximation, $\frac{1}{A}$, and the mean of the true surge distribution coincide exactly, the reciprocal of the mean obtained from equation (1) can be substituted for A in equation (14) and the problem is solved.

For example, if the surge distribution of an exponential process is desired, examination of equations (3) and (14) effortlessly give

$$\hat{p}_\tau(\alpha) = \frac{-R''(0)}{8} \exp[- \frac{-R''(0)}{8} \alpha] \tag{17}$$

It is interesting to note that this approximation to the surge distribution of an exponential process is independent of the surge level x_o.

Characteristics of the New Approximation

The key to understanding Denisenko's approximation to the surge distribution lies in understanding the implications of equations (8) and (9), as these are the only approximations made in the derivation of the estimate, equation (14), from the exact expression, equation (6).

In Denisenko's paper one of the basic premises, rewritten here as equation (8), was given without proof.

$$CD1 \geq CD2$$

where

$$CD1 = \text{conditional density } 1 = P(X(t_k) > x_o | X(t_{k-1}) > x_o)$$

and

$$CD2 = \text{conditional density } 2$$

$$= P(X(t_k) > x_o | X(t_1) > x_o, X(t_2) > x_o, \ldots X(t_{k-1}) > x_o)$$

Critical scrutiny shows this claim to be false. A counter-example is outlined in Appendix A where it is shown that CD1 can also be less than CD2. This leaves no particular magnitude

relationship between CD1 and CD2. If equation (8) were true, then the approximation of equation (9) CD1 = CD2 would introduce a consistent bias into the estimate of the surge distribution, <u>i.e.</u> the estimated cumulative distribution function would always be greater than the actual cumulative distribution function. Because equation (8) fails, no consistent bias should be expected in the estimate of the surge distribution.

$$P(X(t_k) > x_o | X(t_1) > x_o, X(t_2) > x_o, \ldots X(t_{k-1}) > x_o)$$

$$\simeq P(X(t_k) > x_o | X(t_{k-1}) > x_o)$$

If all of the greater than signs in the conditions of equation (9) are replaced with equal signs, then equation (9) would be true for a Markov process. If $X(t)$ is a normal process and Markov, then by Doob's Theorem, $X(t)$ has an exponential correlation function, $R(\tau)$.[8] *Thus if Denisenko's estimate* is used to approximate the surge distribution of a normal process with exponential correlation function, an estimate which is reasonably close to the true distribution would be expected and inaccuracies in the approximation would be due to the greater than signs instead of equal signs in equation (9). If Beckmann's extension is used for non-normal processes with exponential correlation functions, further inaccuracies should be

[8] J. L. Doob. <u>Stochastic Processes</u>. New York, John Wiley & Sons, Inc., 1953, p. 233. This result, commonly known as "Doob's Theorem", was never stated as a theorem by Doob. He gave this result for normal processes as a special case of a much more general theorem on correlation functions of Markov processes.

expected because Doob's Theorem no longer applies for the non-normal cases, unless they are analogus to a normal process (see footnote 3 of this chapter). If a process Y(t) is obtained as a non-linear mapping of a normal process, as was done in this case, then Y(t) is not analogus to normal.

If the approximation is applied to a process with arbitrary correlation function, the accuracy of the approximation would be very sensitive to how well equation (9) holds for the particular correlation function in question.

CHAPTER III

EXPERIMENTAL PROCEDURE

To examine Denisenko's approximation to the surge distribution, a computer program was used which models the process X(t) by a digital Monte Carlo simulation and records the duration of surges above the level x_o. The program is capable of simulating X(t) as either a normal process or an exponential process which allows evaluation of Beckmann's extension to non-Gaussian processes as well as the estimate of Gaussian processes.

The Second Order Autoregressive Process

As indicated at the end of Chapter II, it is necessary to evaluate the performance of the estimators for processes with many different types of correlation functions. To give the program this flexibility, X(t) was generated using a second order autoregressive process. The discrete second order autoregressive process with zero mean is

$$X_k = A_1 \cdot X_{k-1} + A_2 \cdot X_{k-2} + Z_k \qquad (18)$$

where Z_k is independent of X_k for all k, and Z_k is white. Adjustments of the coefficients A1 and A2 in equation (18) make possible a wide range of autocorrelation functions of X_k.

Generating the Gaussian Process

Because the second order autoregressive process is linear, an input process Z_k which is Gaussian will result in an output process X_k which is also Gaussian.

To achieve the uncorrelated normal process Z_k the so-called "direct method" was used[1]

$$Z_k = \sqrt{-2 \ln U_k} \cos (2\pi U_{k+1})$$

$$Z_{k+1} = \sqrt{-2 \ln U_k} \sin (2\pi U_{k+1})$$

where the varieties U_k are independent, uncorrelated and uniformly distributed on the interval from zero to one. The values of U_k were obtained from the function RANF of the CDC FORTRAN library. This gives Z_k with mean zero and variance one. Because the output of the autoregressive process is also required to have zero mean and unit variance, the output must be scaled as shown in Appendix B.

Beginning in the Steady State

The process generator which results from the considerations of the previous two sections is

$$X_k = K(A_1 X_{k-1} + A_2 X_{k-2} + Z_k)$$

with K the normalization constant given in Appendix B.

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[1]Milton Abramowitz and Irene Stegun, Handbook of Mathematical Functions, National Bureau of Standards, 1964, p. 953.

When the simulation is started, initial values for X_{k-1} and X_{k-2} must be chosen so that the process is in the steady state from the first sample onward.[2] When the generator is in the steady state, the values X_{k-1} and X_{k-2} will be normally distributed with zero mean, unit variance, and a non-zero covariance, R.

In Appendix C it is shown that the proper choice of X^*_{k-1} and X^*_{k-2}, where the astrisks indicate initial values, is given by

$$X^*_{k-1} = Z_1$$

and

$$X^*_{k-2} = RZ_1 + \sqrt{1-R^2}\, Z_2$$

where the values Z_1 and Z_2 are independent samples drawn from a distribution which is normal with zero mean and unit variance.

Generating the Exponential Process

The exponential process was generated by taking a function G of the normal process outlined in the previous sections.

Let the process Y be exponentially distributed

$$p_Y(\gamma) = \frac{1}{\beta} \exp[- \frac{\gamma-\alpha}{\beta}] \quad \text{for} \quad \alpha \leq \gamma < \infty$$

$$= 0 \quad \text{for} \quad \gamma < \alpha$$

---

[2]The process generator being in the steady state means that the process is stationary.

so that the mean of Y is $\alpha+\beta$ and the variance of Y is β^2. The cumulative distribution of Y is

$$P(Y \leq \gamma) = 1 - \exp[-\frac{\gamma-\alpha}{\beta}] \quad \text{for} \quad \alpha \leq \gamma < \infty$$

$$= 0 \qquad \text{for} \quad \gamma < \alpha$$

The cumulative distribution function of the normal process X is the Φ function of equation (13). The function G, computed from (15), is

$$G(x) = \beta[-\ln(\frac{1}{2}\,\text{erfc}\,(\frac{x}{2})) + \alpha] \tag{19}$$

This solves the problem of mapping the normal process X into an exponential process Y. The computation of this function is accomplished as shown in Appendix D.

The Process Probability Density Function

To give a quick visual check of the density of the process X(t), a histogram of X(t) is plotted with the data for each modeling of a process. The number of samples of X(t) used to obtain five thousand surges is printed in the title of the histogram. The height of each rectangle of the histogram is normalized so that the summation of the areas of all of the rectangles equals one.

Estimating the Autocorrelation Function

The estimate of the process autocorrelation function is also plotted with the data for each modeling of a process. To avoid having to store all of the samples of the process, the estimate of the autocorrelation function is based only on the first five

thousand samples of the process.

The estimate of the autocorrelation function $\hat{R}(\tau)$ was obtained using the biased estimator

$$\hat{R}(\tau) = \frac{1}{N\hat{\sigma}^2} \sum_{K=1}^{N-\tau} X_k X_{k-\tau}$$

where

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{K=1}^{N} X_k^2$$

This estimator has the desirable property that the variance of the estimator does not depend on τ. Because $R(\tau)$ was only calculated for twenty five lags and the estimate was based on five thousand samples, the bias in the estimator is not serious.

The Surge Probability Density Function

For every process which was simulated, the actual distribution of the surge lengths $p_\tau(\alpha)$ was approximated by a histogram of the simulation data.

Because the process $X(t)$ was simulated by a discrete process, only surges of integer length could be detected. The rectangles are all centered over integers and the height of the rectangle over integer I is equal to the number of surges of length I, divided by 5000. Thus the width of all rectangles is one and the sum of all rectangle heights equals unity.

<u>Confidence Intervals</u>

The histogram of surge lengths represents an estimate of the true surge length distribution. Because an evaluation of another

estimate (Denisenko's $\hat{p}_\tau(\alpha)$) is made by comparing it to the histogram, it is desirable to have a measure of how accurately the histogram represents the true surge distribution.

To accomplish this, the 5000 surges represented by each histogram were divided into twenty separate epochs of 250 surges each. By examining the variation from epoch to epoch of each rectangle, a 95% confidence interval was calculated for each rectangle. The confidence interval was indicated by tick marks of length one third above and below the top of each rectangle.

In the calculation of the confidence intervals, it was assumed that the heights of the twenty rectangles were normally distributed about the true value, and that the epochs were independent. The first assumption can be partially justified by saying that the heights represent the sums of repeated independent trials. In order to have epochs with low correlation (if not independence), the process was run for one hundred samples after the end of each epoch before starting to take surge data for the next epoch. During this one hundred sample period between epochs data was still taken for the computation of $-R''(0)$ (to be discussed later), the histogram of the process probability density (not to be confused with the surge density), and the autocorrelation estimate.

Because the process $X(t)$ is modeled by the discrete process X_k, certain quantization errors will be present in the surge length histogram. These errors are discussed in Appendix F. These errors make it incorrect to say that, "the true distribution lies within the confidence interval with probability 95%", which would be true if the quantization problem were not present. The confidence

interval is still useful, though, as an indication of the variance of the histogram. If the confidence interval for a given rectangle is small then the height of the rectangle may well be close to the true value of the distribution, and, at worst, it can be said that any errors made were committed in a consistent manner.

Epochs Beginning in Mid-Surge

Breaking up the simulation into epochs introduces the potential for another type of error not considered in Appendix F. If the process X_k is above the surge level x_o when the epoch begins, then the first part of that surge will have been ignored and the resulting surge distribution will have bias.

To avoid this, the program checks X_k at the beginning of each epoch and if it is greater than x_o then the remainder of this surge is ignored and the epoch effectively begins with the first sample of the next surge.

Finding the $-R''(0)$

To calculate Denisenko's estimate of the surge length distribution from equations (11) and (12) or (17), the value of $-R''(0)$ for the process X_k is required.

The autocorrelation function of a second order autoregressive process is known exactly.[3] Let A_1 and A_2 be the coefficients of the second order autoregressive process. If the roots of the

---

[3]Gwilyn M. Jenkins and Donald G. Watts, Spectral Analysis and its Applications. San Francisco, Holden-Day, 1968, p. 166.

equation $y^2 - A_1 y - A_2 = 0$ are π_1 and π_2 and if these roots are real then

$$R(\tau) = \frac{\pi_1(1 - \pi_2^2)\pi_1^{|\tau|} - \pi_2(1 - \pi_1^2)\pi_2^{|\tau|}}{(\pi_1 - \pi_2)(1 + \pi_1\pi_2)\sigma_X^2} \tag{20}$$

and if the roots are complex

$$R(\tau) = \frac{K^{|\tau|}\cos(2\pi f_o \tau - \phi_o)}{\sigma_X^2 \cos\phi_o} \tag{21}$$

where

$$K = \sqrt{-A_2}$$

$$f_o = \frac{1}{2\pi} \text{ arc } \cos\left(\frac{A_1}{2\sqrt{-A_2}}\right)$$

and

$$\phi_o = \text{arc } \tan\left(\frac{1 - K^2}{1 + K^2} \tan 2\pi f_o\right)$$

In theory then, all that needs to be done is evaluating the second derivative with respect to τ of the proper equation ((20) or (21)) and evaluating it at zero. The problem is that the function $R(\tau)$ often has a sharp peak at $\tau = 0$ and thus $R(\tau)$ has no first derivative at zero, and therefore the second derivative does not exist at this point. This problem can sometimes be avoided by defining a new $R(\tau)$ function in the neighborhood of zero with a

"rounded" peak, but the results become very ambiguous.[4]

In the continuous case, the quantity $-R''(0)$ is the expected value of the square of the first derivative of the process. The first derivative of a continuous process corresponds to the first difference of a discrete process. This gives the estimator for $-R''(0)$

$$-R''(0) = \frac{1}{N-1} \sum_{k=2}^{N} (X_k - X_{k-1})^2$$

For N on the order of fifty thousand samples (which was usually the case) this estimator is accurate. To avoid the problem, outlined above, of derivatives not existing, this estimator was used for the value of $-R''(0)$ in the calculation of Denisenko's estimate $\hat{p}_\tau(\alpha)$.

Plotting Denisenko's Estimate

To facilitate easy comparison of Denisenko's estimate, $p_\tau(\alpha)$, with the histogram of the experimental data (a curve as close to the true $p_\tau(\alpha)$ as is possible to obtain) the estimate $\hat{p}_\tau(\alpha)$ is output as a smooth curve on the same plot as the histogram.

Because the process $X(t)$ is modeled by the discrete process X_k, it is impossible to accurately detect surges of length less than one. The histogram rectangle corresponding to surges of length one extends from one half to one and one half. Thus the histogram has

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[4]Petr Beckmann, Probability in Communication Engineering. New York, Harcourt, Brace & World Inc., 1967, pp. 224-226.

all of its area in the region α greater than one half. The estimate $\hat{p}_\tau(\alpha)$ is of the surge length distribution of the continuous process X(t) and thus it has all of its area in the region α greater than zero. If a direct comparison is made between $\hat{p}_\tau(\alpha)$ and the histogram, one would not expect a good fit of the two curves even if both were error free representations of the true distribution. This is because they have differing lower bounds on their definitions.

To avoid this problem, and allow a more accurate evaluation of Denisenko's estimate, the smooth curve which is actually plotted is a scaled version of Denisenko's estimate. Let $\hat{p}^*_\tau(\alpha)$ be this scaled estimate. Then

$$\hat{p}^*_\tau(\alpha) = K\hat{p}_\tau(\alpha)$$

where K is chosen so that

$$\int_{.5}^{\infty} \hat{p}^*_\tau(\alpha)\, d\alpha = 1$$

Thus

$$K = \frac{1}{1 - \int_{0}^{.5} \hat{p}_\tau(\alpha)\, d\alpha}$$

and the curve $\hat{p}^*_\tau(\alpha)$ is only plotted for values of α greater than one half.

The Surge Cumulative Density Function

In addition to the plot of the surge probability density function, a plot of the surge cumulative density function is given with the data for each simulation of a process. The experimental

data is plotted as dots and Denisenko's estimate of the cumulative density is plotted as a smooth curve. As with the PDF estimate, the CDF estimate is scaled and only defined per values greater than one half.

CHAPTER IV

DISCUSSION

The data from eleven different simulations is included in this chapter. All of the data for each simulation is presented in a single figure. The equation in the upper left part of each figure describes the marginal distribution of the process X_k. A_1 and A_2 are the coefficients of the second order autoregressive process (see equation (18)) which was used to generate the process X_k. Because the graphs are reproduced in small scale, the values printed along the axis are difficult to read. For this reason, the extreme values for each axis are typed at both ends. Because all scales are linear, the reader can easily interpolate between the extreme values.

What constitutes a "good" fit is a question which is left entirely up to the eye of the reader and no attempt has been made to define an index of performance such as mean square error or mean absolute error.

Processes With Exponential Correlation Functions

If the coefficient A_2 of the second order autoregressive process is equal to zero and the process is Gaussian, then the correlation function of the process is exponential. Figures 3 and 4 show that the correlation function is also approximately exponential if the process is exponential.

As discussed at the end of Chapter 2, good performance of the estimator should be expected for processes with exponential correlation functions. The fit of the estimate to the data (looking at probability density function plots) in figures 1 through 4 is much better for Gaussian processes than exponential ones. Comparison of figures 1 and 2 as well as 3 and 4 shows that the performance remains roughly the same as the surge level is changed.

Processes With Periodic Components

The processes represented in figures 5, 6 and 7 cause the estimator performance to be worse than it was in the exponential case. Again the fit is better for Gaussian processes than exponential. Comparison of figures 5 and 6 shows again that the performance of the estimator is not strongly affected by changes in the surge level.

The processes of figures 8 and 9 have correlation functions which are very unexponential in character. With these processes, the estimator does not fit the data well at all and does not even have the correct shape. These processes have a band pass spectrum and so, while still random and stationary, they have a definite periodic component. If a process has a periodic component and the surge level is less than the magnitude of this component, then obviously the most probable surge length will not be zero. This intuitive notion is verified by figure 8.

The form of Denisenko's estimator guarantees that the estimate of the surge duration distribution will be exponential, and thus it will always predict that the most likely surge length is zero. With

$$p_X(\alpha) = \frac{1}{\sqrt{2\pi}} \exp - \frac{\alpha^2}{2}$$

$A_1 = .7$

surge level $x_o = .5$

$A_2 = 0$

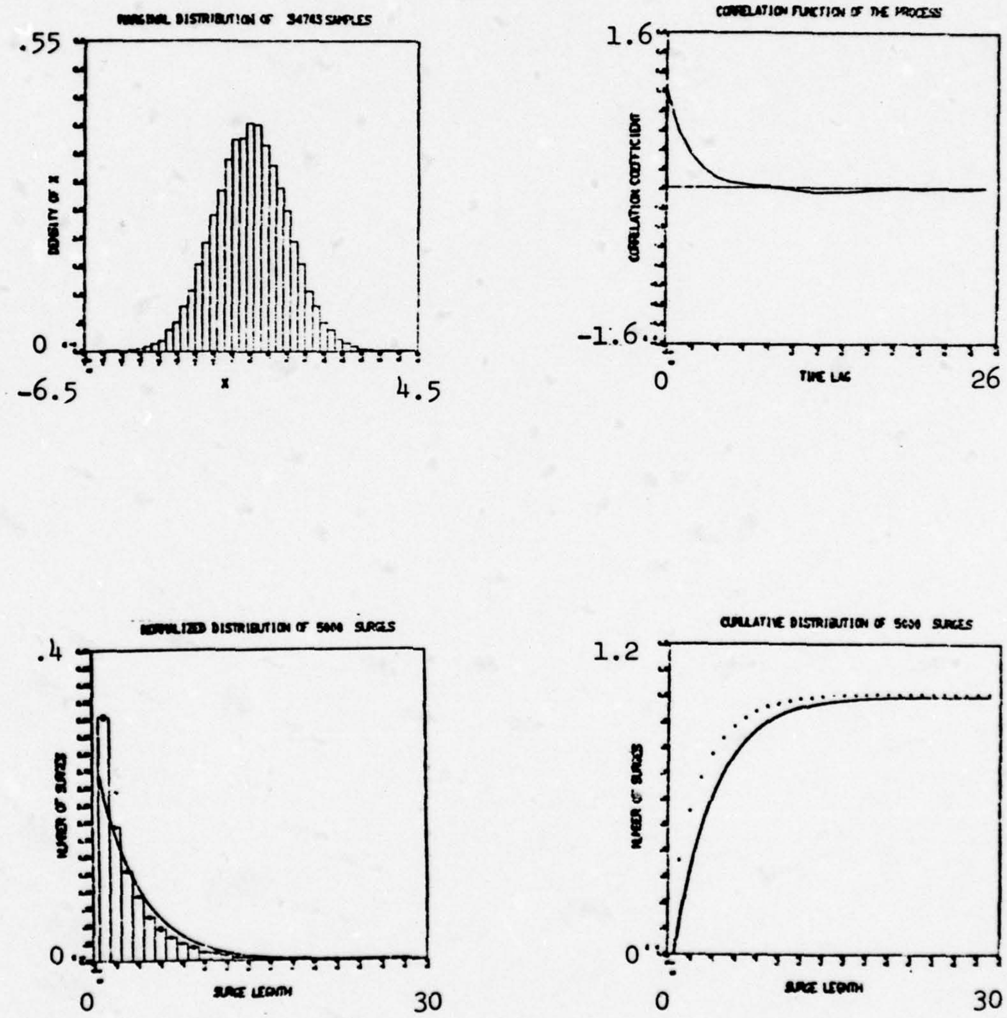

Fig. 1 Data From Simulation No. 1

$$p_X(\alpha) = \frac{1}{\sqrt{2\pi}} \exp - \frac{\alpha^2}{2} \qquad A_1 = .7$$

surge level $x_o = 0$ $\qquad A_2 = 0$

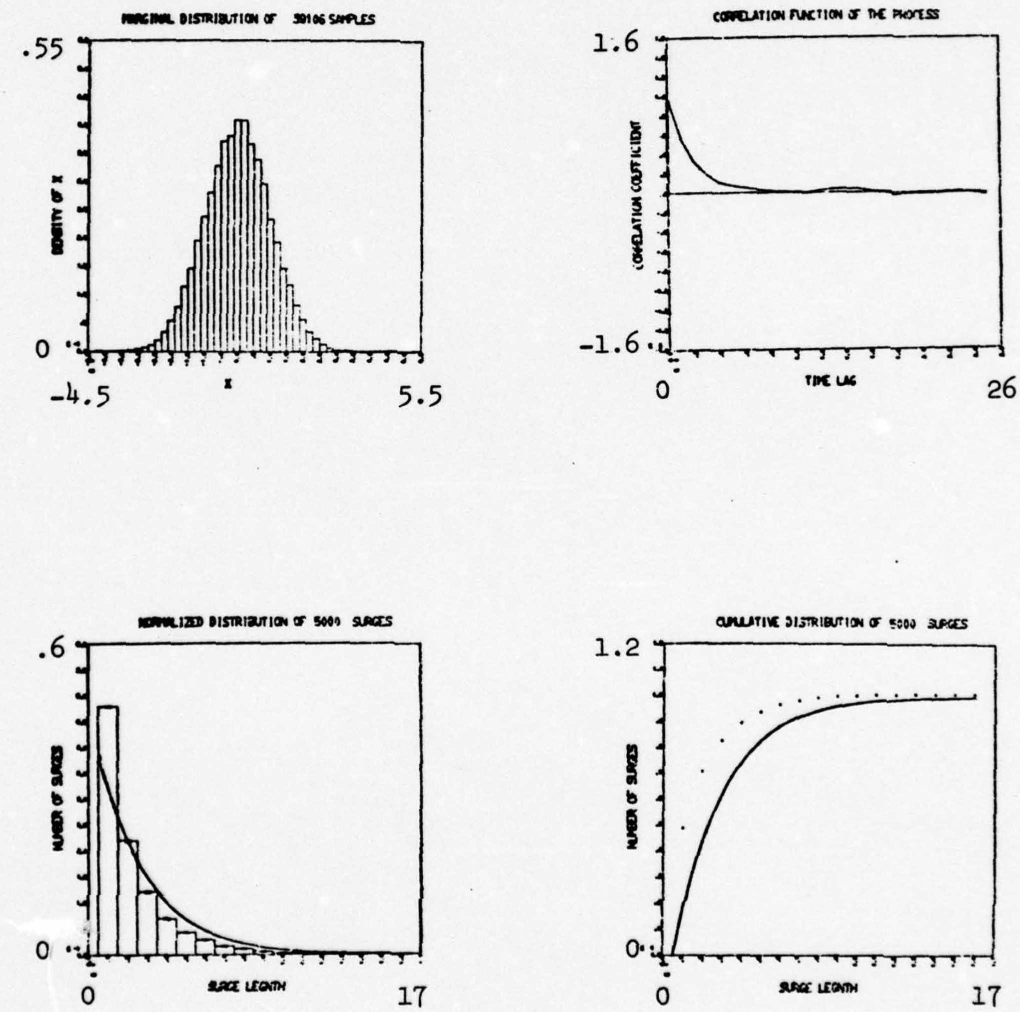

Fig. 2 Data From Simulation No. 2

$p_X(\alpha) = \frac{1}{5} \exp - \frac{\alpha}{5}$

$A_1 = .7$

surge level $x_o = 5$

$A_2 = 0$

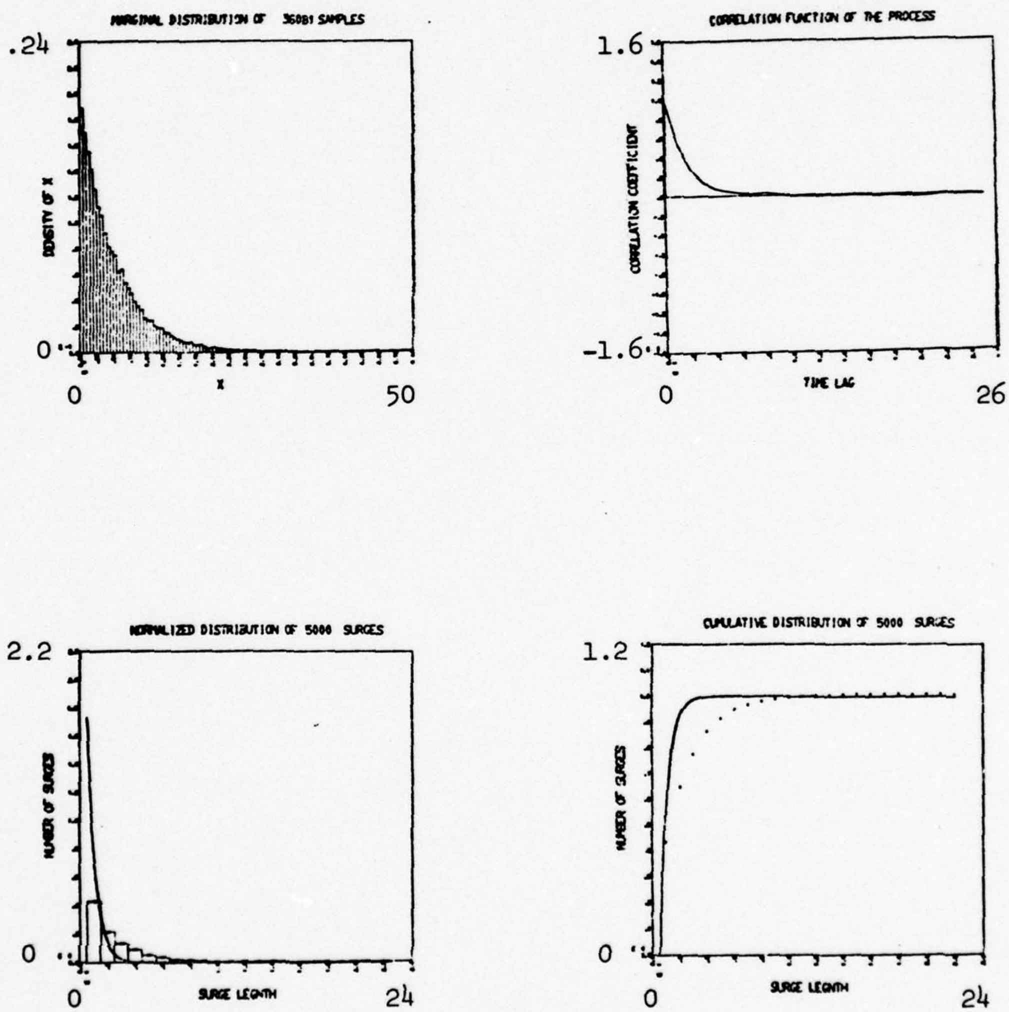

Fig. 3 Data From Simulation No. 3

$p_X(\alpha) = \frac{1}{5} \exp - \frac{\alpha}{5}$ $\qquad$ $A_1 = .7$

surge level $x_o = 10$ $\qquad$ $A_2 = 0$

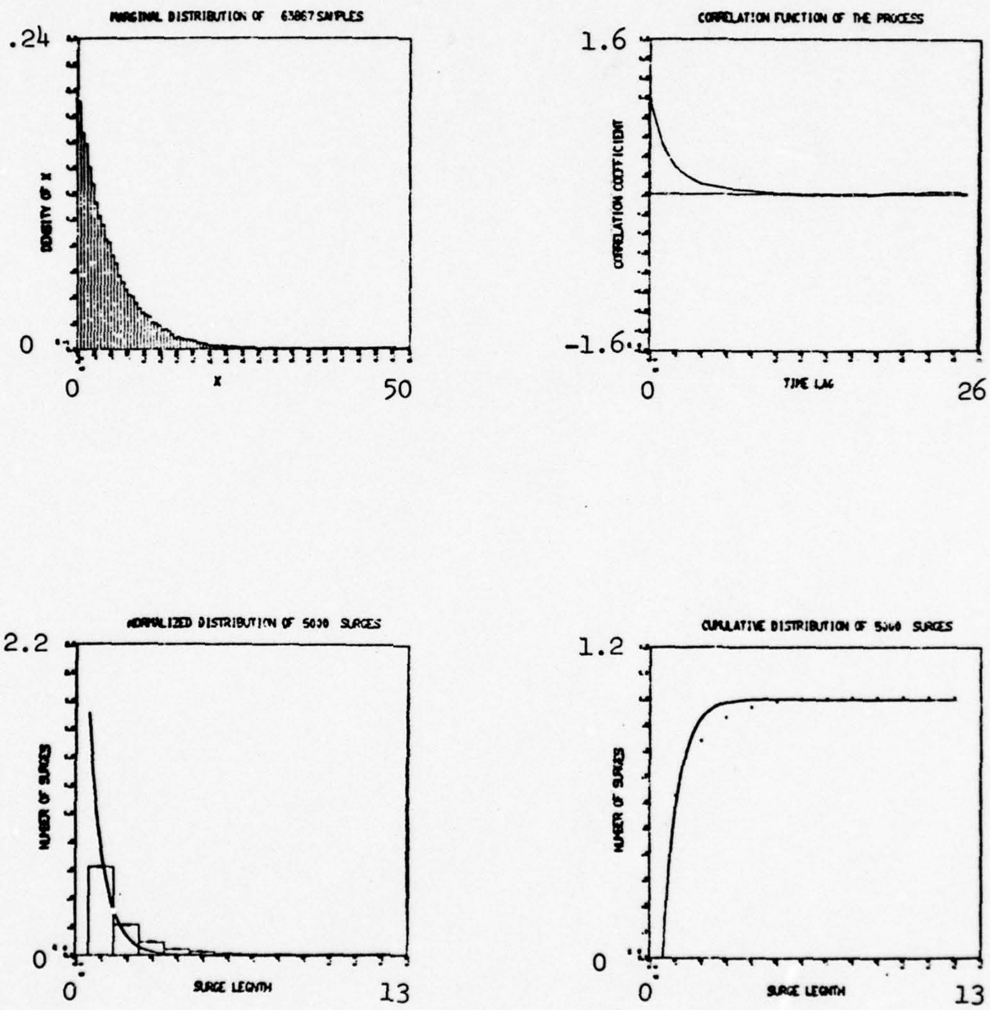

Fig. 4 Data From Simulation No. 4

$$p_X(\alpha) = \frac{1}{\sqrt{2\pi}} \exp - \frac{\alpha^2}{2}$$

$A_1 = 1.7$

surge level $x_o = 0$

$A_2 = -.7025$

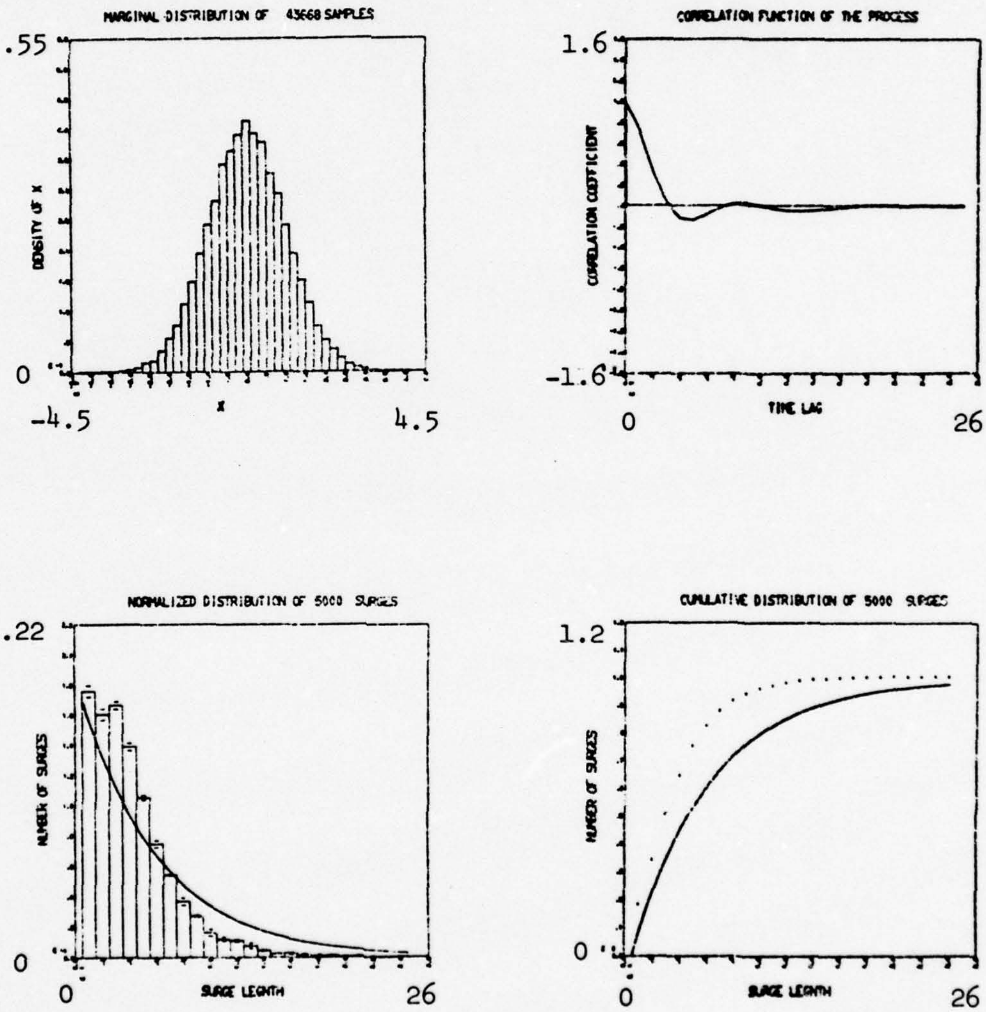

Fig. 5 Data From Simulation No. 5

$p_X(\alpha) = \frac{1}{\sqrt{2\pi}} \exp - \frac{\alpha^2}{2}$ $\qquad A_1 = 1.7$

surge level $x_o = .5$ $\qquad A_2 = -.7025$

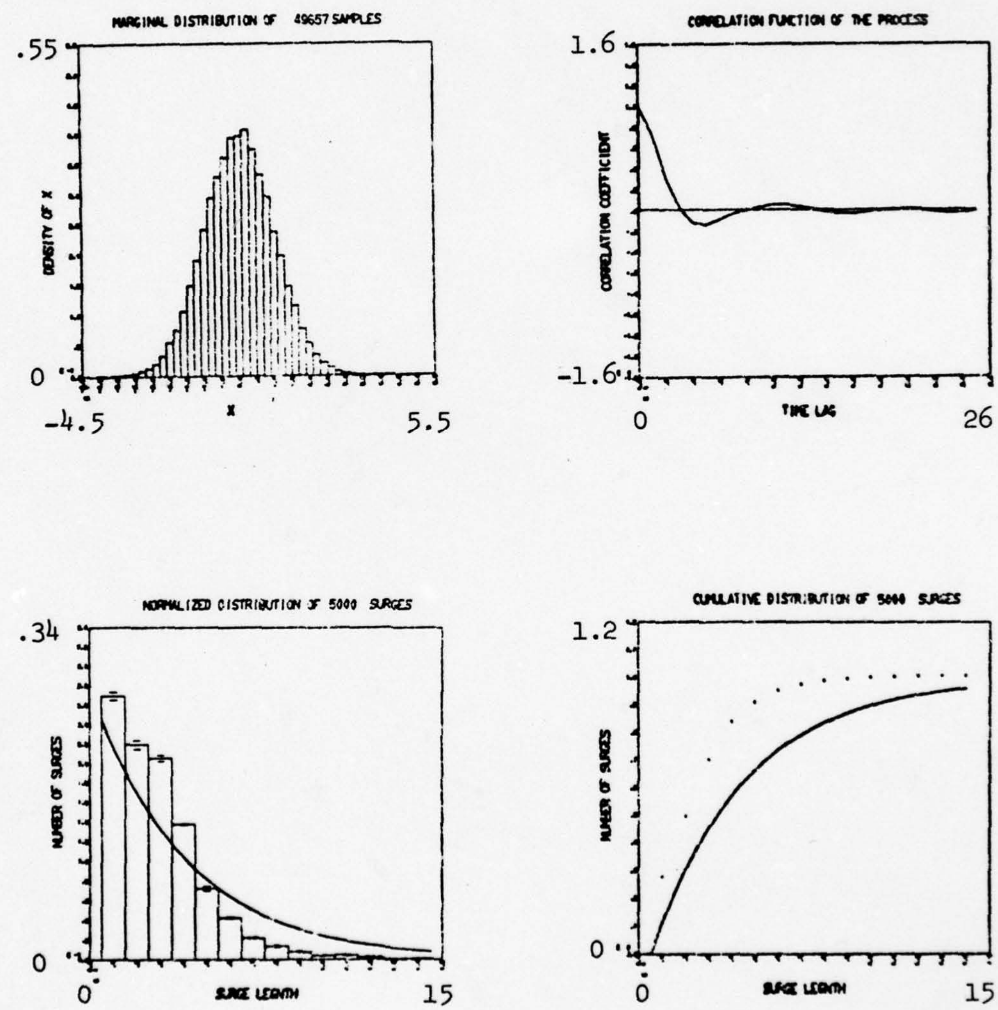

Fig. 6 Data From Simulation No. 6

$p_X(\alpha) = \frac{1}{5} \exp - \frac{\alpha}{5}$ $\qquad\qquad$ $A_1 = 1.7$

surge level $x_o = 7$ $\qquad\qquad$ $A_2 = -.7025$

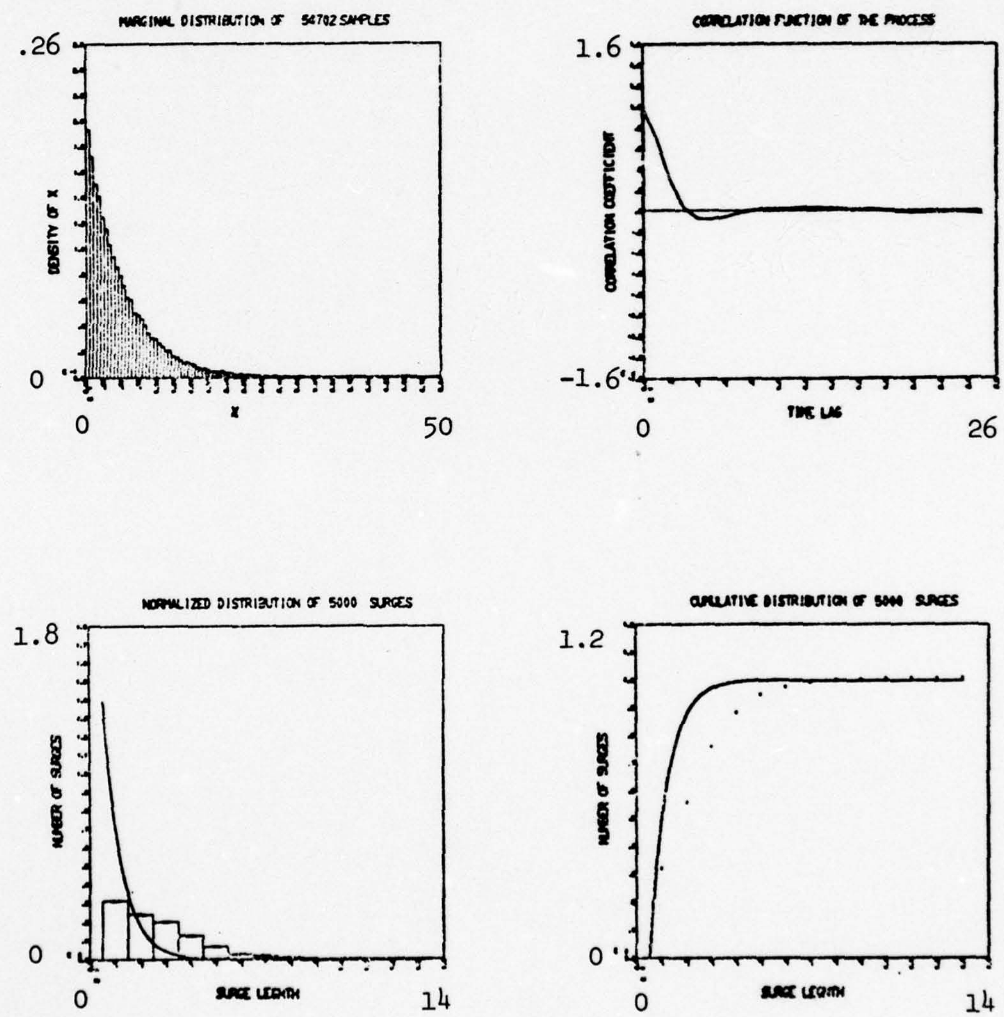

Fig. 7 Data From Simulation No. 7

$$p_X(\alpha) = \frac{1}{\sqrt{2\pi}} \exp - \frac{\alpha^2}{2} \qquad A_1 = 1.995$$

surge level $x_o = .5$ $\qquad A_2 = -.9975$

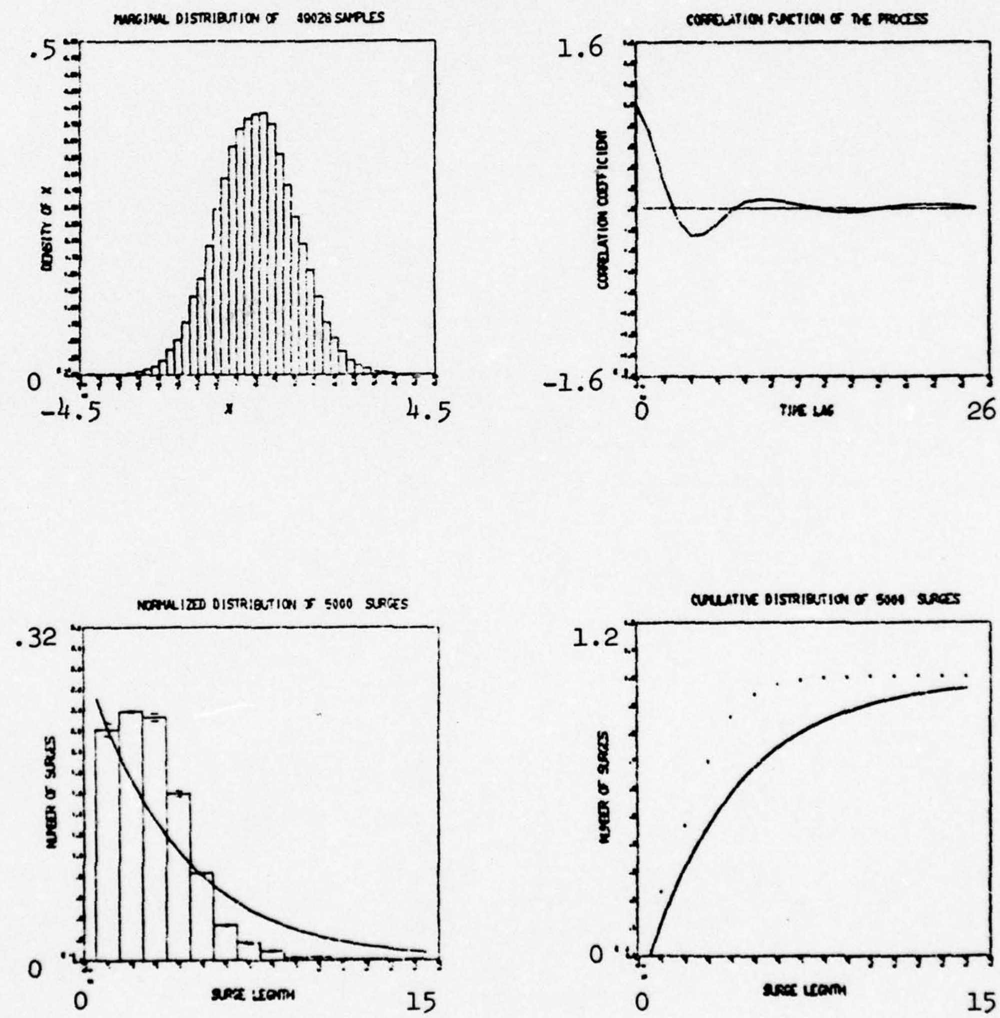

Fig. 8 Data From Simulation No. 8

$p_X(\alpha) = \frac{1}{5} \exp - \frac{\alpha}{5}$ $\qquad A_1 = 1.995$

surge level $x_o = 7$ $\qquad A_2 = -.9975$

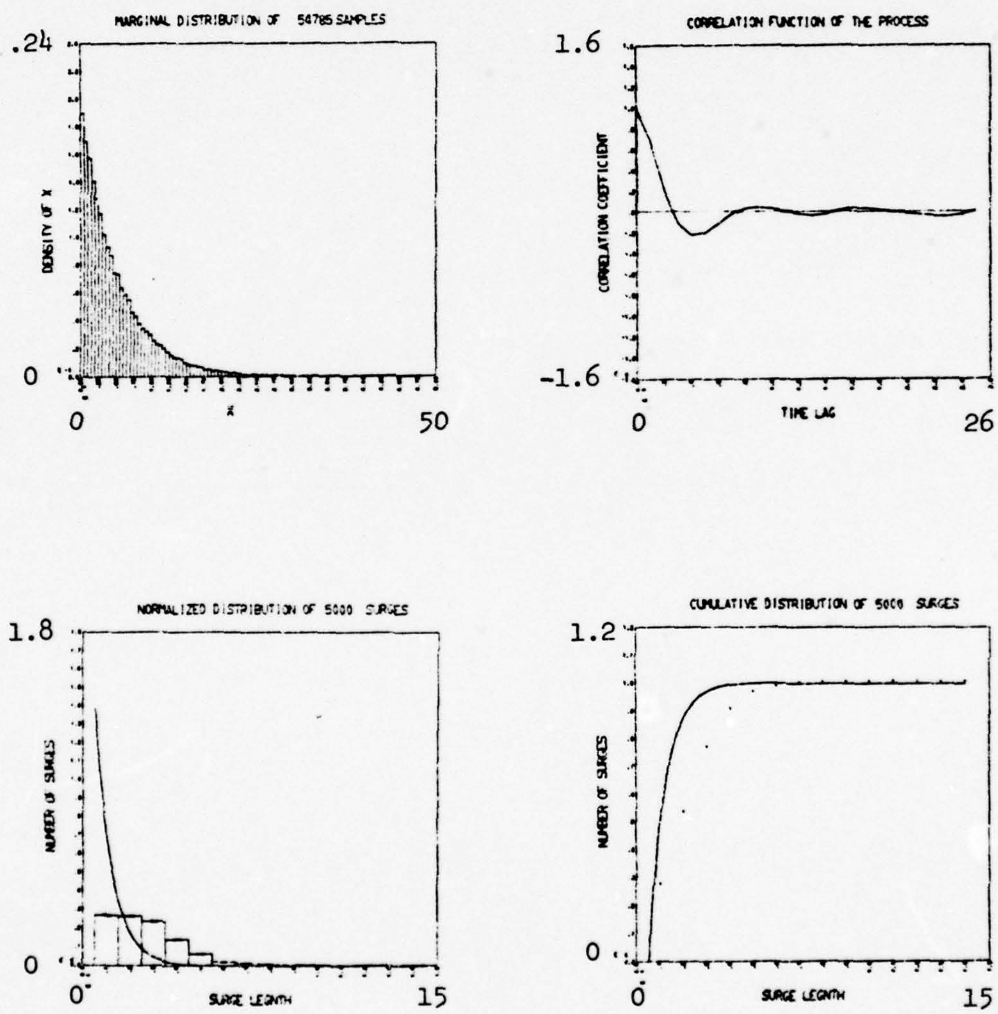

Fig. 9 Data From Simulation No. 9

this in mind, it is not surprising that the estimator does not do a good job with processes having periodic components.

If the process has a periodic component, and the surge level is adjusted so that the most likely surge length is one, then the flaws in the estimator will occur for surge lengths less than one and the quantization discussed earlier will hide these flaws and give a fit which is apparently very good. This is what is happening with the processes in figures 10 and 11. These figures also show that the estimator performance is dependent on the criterion used to evaluate performance. If for a certain application one is concerned with getting a good fit to the cumulative density function of the distribution then the estimator would be said to perform better with the exponential process. If instead, one were interested in a good fit to the probability density function of the distribution then the estimator would be said to perform best with the Gaussian process.

Conclusions

It has been seen that the performance of Denisenko's estimator of the surge length distribution depends a great deal on the correlation function of the process. In general, the estimator improves as the correlation function of the process approaches an exponential function.

It is also concluded that the estimator does not work as well, when applied to non-Gaussian processes. It should be remembered, though, that Rice's estimate does not work at all for non-Gaussian processes, so in this case the extended Denisenko estimator may well

$$p_X(\alpha) = \frac{1}{\sqrt{2\pi}} \exp - \frac{\alpha^2}{2} \qquad A_1 = -.5$$

surge level $x_o = .5$ $\qquad A_2 = -.9975$

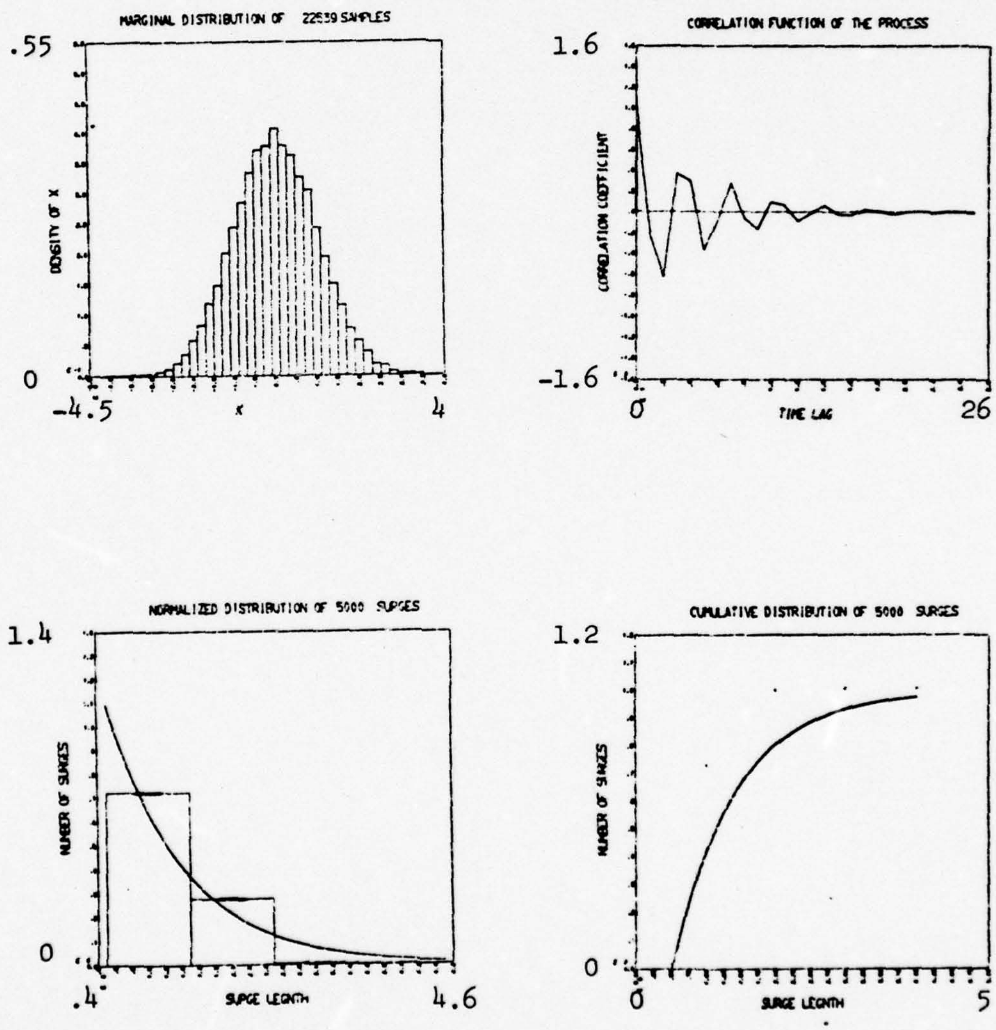

Fig. 10 Data From Simulation No. 10

$p_X(\alpha) = \frac{1}{5} \exp - \frac{\alpha}{5}$

$A_1 = -.5$

surge level $x_o = 7$

$A_2 = -.9975$

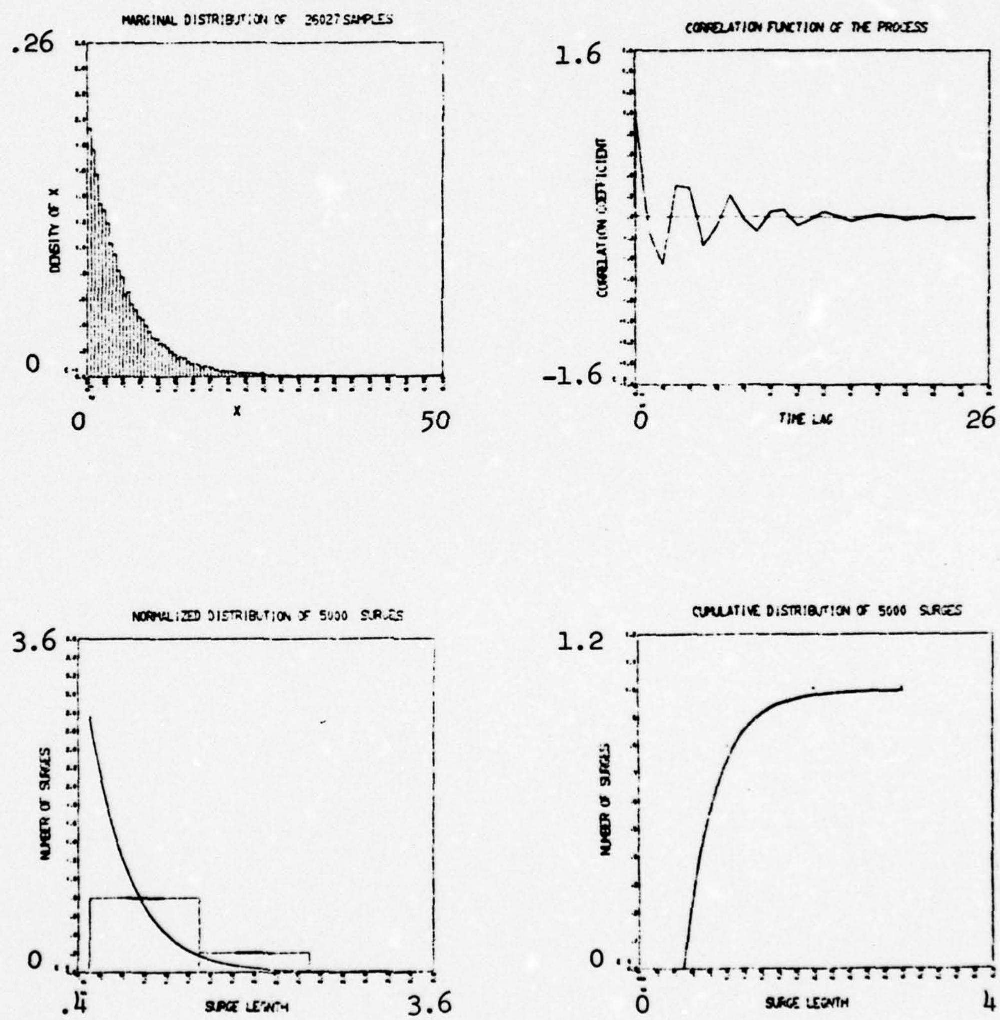

Fig. 11 Data From Simulation No. 11

be as good as one can do.

So finally, if an application requires an estimate of the surge distribution of a normal process, and the correlation function is similar to an exponential function, then Denisenko's estimate may well be good enough, and is far simpler to obtain than Rice's. If the correlation function is not close to exponential and good accuracy of the estimator is required, then Rice's method, first introduced in 1958, still stands as the best solution.

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APPENDICES

APPENDIX A

A COUNTER-EXAMPLE OF THE CLAIM MADE IN EQUATION (8)

Consider the process X(t) which is normally distributed with zero mean and unit variance for all t. Let the autocorrelation series of X(t) be such that

$$R(0) = 1$$

$$R(1) = 0$$

and

$$R(2) = r$$

and consider surges of X(t) above level $x_o = 0$. The claim of equation (8) is:

$$P(X(t) > 0 | X(t-1) > 0) \geq P(X(t) > 0 | X(t-1) > 0,\ X(t-2) > 0) \tag{22}$$

Let $X(t) = x_1$, $X(t-1) = x_2$, and $X(t-2) = x_3$. Now because $COV(x_1,x_2) = 0$ and x_1 and x_2 are both normally distributed, x_1 is independent of x_2. Thus

$$P(x_1 > 0 | x_2 > 0) = P(x_1 > 0)$$

$$= 1 - \Phi(0)$$

$$= .5$$

Now

$$P(x_1 > 0 | x_2 > 0, x_3 > 0) = \frac{P(x_1 > 0, x_2 > 0, x_3 > 0)}{P(x_2 > 0, x_3 > 0)}$$

But again because $R(1) = 0$ and x_2 and x_3 are both normal, x_2 and x_3 are independent.

So

$$P(x_2 > 0, x_3 > 0) = P(x_2 > 0) \cdot P(x_3 > 0)$$

$$= [1 - \Phi(0)][1 - \Phi(0)]$$

$$= \frac{1}{2} \cdot \frac{1}{2}$$

$$= .25$$

Equation (22) now says

$$.5 \geq \frac{P(x_1 > 0, x_2 > 0, x_3 > 0)}{.25}$$

or

$$.125 \geq P(x_1 > 0, x_2 > 0, x_3 > 0) \tag{23}$$

The term on the right is

$$\int_0^\infty \int_0^\infty \int_0^\infty P(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 \tag{24}$$

Using vector notation, the integrand is written as[1]

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[1]Single underlining of variables will represent column vectors, double underlining will represent matrices and a raised T is the transpose operation.

$$P(\underline{X}) = P\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The multivariate normal density with zero mean is

$$P(\underline{X}) = \frac{1}{(2\pi)^{d/2}|\underline{\underline{\Sigma}}|^{1/2}} \exp[-\tfrac{1}{2}\,\underline{X}^T\,\underline{\underline{\Sigma}}^{-1}\,\underline{X}] \tag{25}$$

where d is the dimension and $\underline{\underline{\Sigma}}$ is the covariance matrix.

In this case, d=3 and

$$\underline{\underline{\Sigma}} = \begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ r & 0 & 1 \end{pmatrix}$$

So

$$|\underline{\underline{\Sigma}}| = 1-r^2$$

and

$$\underline{\underline{\Sigma}}^{-1} = \begin{pmatrix} A & 0 & B \\ 0 & 1 & 0 \\ B & 0 & A \end{pmatrix}$$

where $A = \dfrac{1}{1-r^2}$ and $B = -rA$. Now equation (25) gives

$$P(\underline{X}) = \frac{1}{(2\ )^{3/2}\sqrt{1-r^2}} \exp -\tfrac{1}{2}\,(x_1,x_2,x_3) \begin{pmatrix} A & 0 & B \\ 0 & 1 & 0 \\ B & 0 & A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Evaluation of the product in the argument of the exponential gives

$$\frac{1}{2}(Ax_1^2 + 2Bx_1x_3 + Ax_3^2 + x_2^2)$$

Returning to scalar notation and substituting into equation (24) yields

$$P(x_1 > 0,\ x_2 > 0,\ x_3 > 0)$$

$$= \int_o^\infty\int_o^\infty\int_o^\infty \frac{1}{(2\pi)^{3/2}\sqrt{1-r^2}} \exp\left[-\frac{Ax_1^2+2Bx_1x_3+Ax_3^2+x_2^2}{2}\right] dx_1\ dx_2\ dx_3$$

$$= \frac{1}{(2\pi)^{3/2}\sqrt{1-r^2}} \int_o^\infty\int_o^\infty \exp\ [-\frac{Ax_1^2+2Bx_1x_3+Ax_2^2}{1}] \int_o^\infty \exp[-\frac{x_2^2}{1}]dx_2dx_3dx_1$$

Using

$$\int_o^\infty \exp(-\frac{t^2}{2})dt = \frac{\sqrt{\pi}}{\sqrt{2}} \qquad (26)$$

to evaluate the inner integral yields

$$P(x_1 > 0,\ x_2 > 0,\ x_3 > 0)$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}\ (2\pi)^{3/2}\sqrt{1-r^2}} \int_o^\infty \exp(-\frac{Ax_1^2}{2}) \int_o^\infty \exp\ [-\frac{2Bx_1+x_3+Ax_3^2}{2}]\ dx_3\ dx_1$$

completing the square in the argument of the second exponential yields

$$P(x_1 > 0, x_2 > 0, x_3 > 0)$$

$$= K_1 \int_0^\infty \exp\left[-\frac{Ax_1^2}{2}\right] \exp\left[\frac{B^2 x_1^2}{2\,A}\right] \int_0^\infty \exp\left[-\frac{\left(\sqrt{A}\, x_3 + \frac{Bx_1}{\sqrt{A}}\right)^2}{2}\right] dx_3\, dx_1$$

where

$$K_1 = \frac{\sqrt{\pi}}{\sqrt{2}\, (2\pi)^{3/2} \sqrt{1-r^2}}$$

The change of variable in the inner integral

$$z = \sqrt{A}\, x_3 + \frac{Bx_1}{\sqrt{A}}$$

gives

$$P(x_1 > 0, x_2 > 0, x_3 > 0)$$

$$= \frac{K_1}{\sqrt{A}} \int_0^\infty \exp - \frac{\left(A - \frac{B^2}{\sqrt{A}}\right) x_1^2}{2} \int_{\frac{Bx_1}{\sqrt{A}}}^\infty \exp\left(-\frac{z^2}{2}\right) dz\, dx_1$$

Using the fact that

$$\int_z^\infty \exp\left(-\frac{t^2}{2}\right) dt = \frac{\sqrt{\pi}}{\sqrt{2}} \left(1 - \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right)$$

gives

$$P(x_1 > 0,\ x_2 > 0,\ x_3 > 0)$$

$$= \frac{\pi}{2(2\pi)^{3/2}} \int_0^\infty (1 - \mathrm{erf}(\frac{B}{\sqrt{2A}}\, x_1)\ \exp[- \frac{A\sqrt{A} - B^2}{2\sqrt{A}}\, x_1^2]\ dx_1$$

and the change of variable

$$z = \frac{x_1}{\sqrt{2A}}$$

yields

$$P(x_1 > 0,\ x_2 > 0,\ x_3 > 0)$$

$$= \frac{1}{4} \sqrt{\frac{A}{\pi}} \int_0^\infty (1 - \mathrm{erf}(Bz))\ \exp(- (A^2 - B^2\ A)z^2)\ dz$$

which is this standard form from a table of integrals[2]

$$\int_0^\infty (1 - \mathrm{erf}(\beta x))\ \exp(\mu^2 x^2)\ dx = \frac{1}{2\pi\beta z}\ \ln(\frac{1+z}{1-z})$$

where

$$z = \sqrt{\frac{\mu^2}{\beta^2}} \qquad \text{and} \qquad \beta = B$$

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[2] I. S. Gradshteyn and I. M. Ryzhik. <u>Table of Integrals, Series, and Products</u>. 4th ed., New York, Academic Press, 1965, p. 649 and p. 1041. The pertinent equations are 6.286(1) and 9.121(7).

So finally

$$P(x_1 > 0,\ x_2 > 0,\ x_3 > 0) = \frac{A}{8B\pi Z} \ln(Z)$$

where

$$Z = \frac{1+z}{1-z}$$

Evaluating this expression for r in the autocorrelation sequence equal to .36 gives[3]

$$P(x_1 > 0,\ x_2 > 0,\ x_3 > 0) = .16949$$

and recalling equation (23) gives the contradiction

$$.125 \geq .16949$$

The value .169 is not surprising because .125 is the probability of all three variates being greater than zero if they are all independent, but in this example, x_1 and x_3 are positively correlated.

In order for this to be a valid counter-example, a process with autocorrelation sequence $R(0) = 1$, $R(1) = 0$, and $R(2) = r$ must have an extended autocorrelation sequence which results in a positive definite matrix. As proved by Burg, the only requirement for this to be so is that the three by three matrix with the above values be

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[3]The integral was first evaluated by a computer program using trapazoid rule integration from zero to twenty. This gave the result $P(x_1 > 0,\ x_2 > 0,\ x_3 > 0) = .155$, and the above calculations are a verification of this.

positive definite.[4] For r less than one, the three by three matrix is positive definite so this is a valid counter-example.

[4]John Parker Burg. Maximum Entropy Spectral Analysis. A Dissertation Submitted to the Department of Geophysics and the Committee of Graduate Studies of Stanford University in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy, May 1975, p. 22.

APPENDIX B

NORMALIZATION OF THE AUTOREGRESSIVE PROCESS

Let Y_k be the unnormalized output of the second order autoregressive process.

$$Y_k = A_1 X_{k-1} + A_2 X_{k-2} + Z_k$$

where Z_k are successive samples drawn independetly from a normal distribution with zero mean and unit variance. Z_k is independent of X_j for all k and j. The variance of Y_k is a function of A_1 and A_2 and will not, in general be unity. Thus if $X_k = Y_k$ the process X_k will not be stationary. The task is to find a normalization constant C such that when $X_k = CY_k$, X_k has unit variance for all k.

$$X_k = C(A_1 A_{k-1} + A_2 X_{k-2} + Z_k)$$

and C will be found such that $E[X_k{}^2] = 1$. Expanding the square of X_k gives

$$E[X_k{}^2] = 1 = C^2 E[A_1^2 X_{k-1}^2 + A_2^2 X_{k-2}^2 + 2A_1 A_2 X_{k-1} X_{k-2} + 2A_1 X_{k-1} Z_k$$

$$+ 2A_2 X_{k-2} Z_k + Z_k^2]$$

Collecting terms and evaluating the expected values gives

$$1 = C^2(A_1^2 + A_2^2 + 2A_1 A_2 E[X_{k-1} X_{k-2}] + 1) \tag{27}$$

where

$$E[X_{k-1}X_{k-2}] = E[C(A_1X_{k-2}A_2X_{k-3}Z_{k-1})X_{k-2}]$$

$$= CA_1 + CA_2E[X_{k-1}X_{k-2}]$$

Using the stationarity of X_k gives

$$E[X_{k-1}X_{k-2}] = CA_1 + CA_2E[X_{k-1}X_{k-2}]$$

$$= \frac{CA_1}{1-CA_2} \qquad (28)$$

Substituting this result into equation (27) yields

$$1 = C^2(A_1^2 + A_2^2 + \frac{2A_1A_2C}{1-CA_2} + 1)$$

or

$$1 - CA_2 = C^2A_1^2(1-CA_2) + A_2^2(1-CA_2) + 2A_1^2A_2^2C + 1 - CA_2$$

Regrouping like powers of C gives

$$0 = C^3(A_1^2A_2 - A_2^2 - A_2) + C^2(A_1^2 + A_2^2 + 1) + CA_2 - 1$$

The process generating subroutine solves this cubic equation for the three roots of C, picks the smallest positive real root and uses this value to normalize the process.

APPENDIX C

INITIAL VALUES OF THE AUTOREGRESSIVE PROCESS

Often when generating an autoregressive process, one sets the initial values equal to zero. This gives a process which is not in steady state from the beginning, but has a start up transient. Instead, the initial values X^*_{k-1} and X^*_{k-2} of the second order autoregressive process should be chosen according to the formulas

$$X^*_{k-1} = Z_1 \tag{29}$$

and

$$X^*_{k-2} = RZ_1 + \sqrt{1-R^2}\, Z_2 \tag{30}$$

where the values Z_1 and Z_2 are independent samples from a normal distribution with zero mean and unit variance. R is the covariance of X_{k-1} and X_{k-2} which was shown in equation (28) to be

$$R = \frac{KA_1}{1-KA_2}$$

By inspection, X^*_{k-1} and X^*_{k-2} have zero mean, and X^*_{k-1} has unit variance. The variance of X^*_{k-2} is

$$Var(X^*_{k-2}) = E[X^{*2}_{k-2}] - (E[X^*_{k-2}])^2$$

$$= E[(RZ_1 + \sqrt{1-R^2}\, Z_2)^2] - (E[RZ_1 + \sqrt{1-R^2}\, Z_2])^2$$

$$= E[R^2Z_1^2 + 2RZ_1\sqrt{1-R^2}\, Z_2 + (1-R^2)Z_2^2] - (RE[Z_1^2] + \sqrt{1-R^2}\, E[Z_2])^2$$

$$= R^2E[Z_1^2] + 2R\sqrt{1-R^2}\, E[Z_1Z_2] + (1-R^2)E[Z_2^2]$$

$$= R^2 + (1-R^2)$$

$$= 1$$

So the variance of X^*_{k-2} is correct.

The covariance of X^*_{k-1} and X^*_{k-2} is

$$COV(X^*_{k-1},X^*_{k-2}) = E[X^*_{k-1},X^*_{k-2}] - E[X^*_{k-1}]E[X^*_{k-2}]$$

$$= E[Z_1(RZ_1 + \sqrt{1-R^2}\, Z_2)]$$

$$= E[RZ_1^2 + \sqrt{1-R^2}\, Z_2Z_1]$$

$$= RE[Z_1^2] + \sqrt{1-R^2}\, E[Z_1Z_2]$$

$$= R$$

Thus it has been shown that if X^*_{k-1} and X^*_{k-2} are picked according to equations (29) and (30) then they have the proper means, the proper variances, and the correct covariance.

APPENDIX D

COMPUTATION OF THE MAPPING FUNCTION G(x)

Equation (19) gives the mapping of a normal process into an exponential process. The computation of this function was required every time that an exponential variate was needed. The program generated about 50,000 exponential variates for every simulation, so an efficient way of computing the error function was required. The error function was approximated in the following way.[1]

$$\mathrm{erf}(x) = 1 - (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5)e^{-x^2} + \varepsilon(x)$$

where

$$t = \frac{1}{1+px}$$

$$p = .3275911$$

$$a_1 = .254829592$$

$$a_2 = -.284496736$$

$$a_3 = 1.421413741$$

$$a_4 = -1.453152027$$

$$a_5 = 1.061405429$$

and the magnitude of the error $\varepsilon(x)$ is less than $1.5 \cdot 10^{-7}$.

---

[1]Milton Abramowitz and Irene Stegun. Handbook of Mathematical Functions, National Bureau of Standards, 1964, p. 299.

APPENDIX E

THE MEAN OF DENISENKO'S ESTIMATE

Equation (14) gives Denisenko's approximation to the surge length distribution as

$$\hat{p}_\tau(\alpha) = Ae^{-A\alpha}$$

where

$$A(x_o) = \frac{\sqrt{-R''(0)}}{2\left[1 - \Phi\frac{(x_o - \mu)}{\sigma}\right]} \exp\left[-\frac{(x_o - \mu)^2}{2\sigma^2}\right]$$

and

$$\Phi(z) = \frac{1}{2\pi}\int_{-\infty}^{z} e^{-t^2/2}\, dt$$

So

$$E[\tau] = \frac{1}{A}$$

$$= \frac{2\pi\left[1 - \Phi\left(\frac{x_o - \mu}{\sigma}\right)\right]}{\sqrt{-R''(0)}} \exp\left[+\frac{(x_o - \mu)^2}{2\sigma^2}\right]$$

$$= \frac{\pi}{\sqrt{-R''(0)}} \exp\left[\frac{(x_o - \mu)^2}{2\sigma^2}\right] \left\{2\left[1 - \Phi\left(\frac{x_o - \mu}{\sigma}\right)\right]\right\} \qquad (31)$$

With some algebra, the function Φ can be expressed in terms of the error function:

$$\Phi(z) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)$$

substituting this into (31) yields

$$E[\hat{p}_\tau(\alpha)] = \frac{\pi}{\sqrt{-R''(0)}} \exp\left[\frac{(x_o - \mu)^2}{2\sigma^2}\right] \{2[1 - \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\,(\frac{x_o - \mu}{\sigma\sqrt{2}})]\}$$

or

$$E[\hat{p}_\tau(\alpha)] = \frac{\pi}{\sqrt{-R''(0)}} \exp\left[\frac{(x_o - \mu)^2}{2\sigma^2}\right] \{1 - \mathrm{erf}\,(\frac{x_o - \mu}{\sigma\sqrt{2}})\} \qquad (32)$$

And now for comparison, the exact mean, equation (2), is rewritten as follows

$$E[p_\tau(\alpha)] = \frac{\pi}{\sqrt{-R''(0)}} \exp\left[\frac{(x_o - \mu)^2}{2\sigma^2}\right] \{\mathrm{erfc}\,(\frac{x_o - \mu}{\sigma\sqrt{2}})\}$$

or

$$E[p_\tau(\alpha)] = \frac{\pi}{\sqrt{-R''(0)}} \exp\left[\frac{(x_o - \mu)^2}{2\sigma^2}\right] \{1 - \mathrm{erf}\,(\frac{x_o - \mu}{\sigma\sqrt{2}})\}$$

Comparison of this expression with equation (32) shows that Denisenko's approximation gives an unbiased estimate of the mean.

APPENDIX F

ERRORS DUE TO QUANTIZATION

The process $X(t)$ is modeled by the discrete process X_k. X_k can be thought of as the sampling of $X(t)$ at integer time increments. A possible segment of the process $X(t)$ is shown in figure 12 where tick marks indicate sampling times and the axis is the surge threshold. The following list details some of the possible errors due to quantization. Numbers refer to the numbers on figure 12.

1. A short surge is ignored.
2. A surge of length .95 is ignored.
3. A short surge is recorded as having length one.
4. A surge of length 1.9 is recorded as having length one.
5. A surge of length .5 is ignored.
6. Three separate surges of lengths 2.5, 3.5, and 2 are recorded as a single surge of length nine.
7. A series of short surges is ignored.
8. A series of short surges is recorded as a single surge of length four.

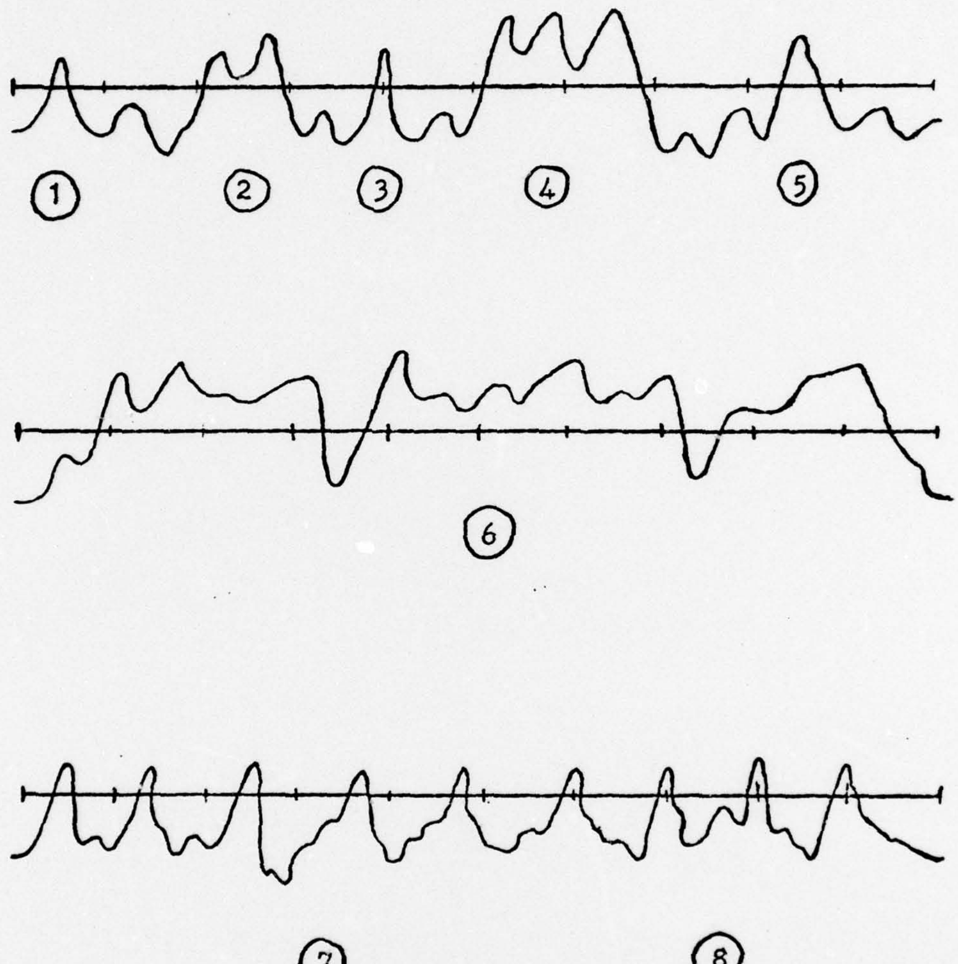

Fig. 12 One Possible Segment of X(t)

APPENDIX G

THE COMPUTER PROGRAM

The process modeling, the calculations of the estimators, and the output of data were all done with a FORTRAN program written for the CDC "RUN" compilor. All graphical output is done with the CDC 280 microfilm plotter, using subroutines in the CUGLIB library.

In addition to the four plots shown in figures one through eleven, the program outputs an additional plot, an example of which is given in figure 13. This plot is for the data of simulation number six. The plot is the cumulative distribution function subject to the mapping

$$X^* = \frac{1}{A} \ln \frac{K}{1-X}$$

where A is given in equation (10) of the text, K is the normalization constant derived in Appendix B, X is a point of the cumulative distribution function and X* is the mapped point which is plotted.

The nature of this mapping is such that Denisenko's estimate becomes the straight line y = x. Thus to see how well the experimental data fits the estimate, one looks at how straight a curve through the points is. The line y = x is also plotted for reference. It should be pointed out that no additional information is contained in these plots which is not already in the regular cumulative density function. For this reason, it is not included

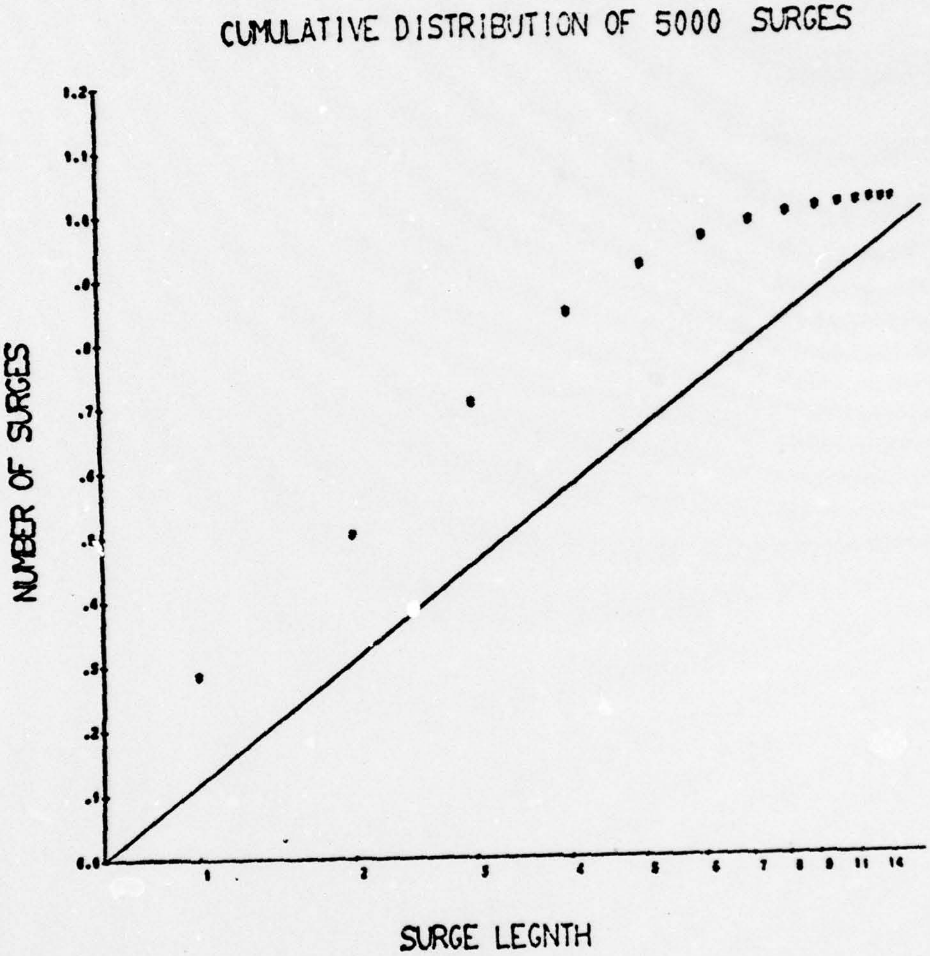

Fig. 13 An Example of a CDF Subject to a Linearizing Mapping

with the regular data in figures one through eleven, but it does give a quick indication of how well the estimator performs for a given process and the routine has been left in the program.

Also included in the program are subroutines which separate on the microfilm the plots of one simulation from those of the preceding simulation as well as that following.

Each simulation requires one data card which contains the surge level in columns two through eleven, the coefficients of the autoregressive process (A_1 in columns twelve through twenty-one and A_2 in columns twenty-two through thirty-one) and a one in column one for a normal process or a zero in column one for an exponential process. Preceding these cards is a seed card for the random number generator which is any number in octal twenty format. The last card in the deck should have a single nine in column one.

```
      PROGRAM SERIES (INPUT,OUTPUT,FILMPL,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION DMAR(100),SURG(20,250),OUTPUT(5000)
      READ(5,1) SEED
    1 FORMAT (O20)
      Q=RANF(SEED)
      DO 2 I=1,100
      READ(5,3) IFNORM,SLEV,A1,A2
3     FORMAT(I1,3F10.4)
      IF(IFNORM.EQ.9) GO TO 4
      CALL CMINIT(0.)
      CALL INCISP
      CALL FRAME
      CALL ARROWD
      CALL OMAIN(A1,A2,SLEV,DMAR,SURG,OUTPUT,IFNORM)
2     CONTINUE
4     CONTINUE
      STOP
      END
```

```
      SUBROUTINE ARROWU(A1,A2,SLEV,IFNORM)
      CALL INCISP
      CALL MAP(0.,1.,0.,1.,0.,1.,0.,1.)
      CALL SETLINE (1)
      XSHFT=0.
    5 CONTINUE
      DO 1 I=1,3
      RI=I
      X=.135-.006*(RI-1.)+XSHFT
      YSTOP=.897-.006*(RI-1.)
    1 CALL LYNE(X,.05,X,YSTOP)
      DO 2 I=1,3
      RI=I
       X=.141+.006*(RI-1.)+XSHFT
      YSTOP=.897-.006*(RI-1.)
    2 CALL LYNE(X,.05,X,YSTOP)
      DO 3 I=1,6
      RI=I
      YTOP=.9+.006*(RI-1.)
      YBTM=YTOP-.138
      XLFT=XSHFT
      XMID=.138+XSHFT
      XRHT=.276+XSHFT
      CALL LYNE(XRHT,YBTM,XMID,YTOP)
3     CALL LYNE(XMID,YTOP,XLFT,YBTM)
      IF(XSHFT.NE.0) GO TO 4
      XSHFT=.724
      GO TO 5
    4 CONTINUE
      CALL LABPLT (A1,A2,SLEV,IFNORM)
      RETURN
      END
```

```
      SUBROUTINE ARROWD
      CALL INCISP
      CALL MAP(0.,1.,0.,1.,0.,1.,0.,1.)
      CALL SETLINE(1)
      XSHFT=0.
    5 CONTINUE
      DO 1 I=1,3
      RI=I
      X=.135-.006*(RI-1.)+XSHFT
      YSTOP=.103+.006*(RI-1.)
    1 CALL LYNE(X,.95,X,YSTOP)
      DO 2 I=1,3
      RI=I
      X=.141+.006*(RI-1.)+XSHFT
      YSTOP=.103+.006*(RI-1.)
    2 CALL LYNE(X,.95,X,YSTOP)
      DO 3 I=1,6
      RI=I
      YBTM=.1-.006*(RI-1.)
      YTOP=YBTM+.138
      XLFT=XSHFT
      XMID=.138+XSHFT
      XRHT=.276+XSHFT
      CALL LYNE(XRHT,YTOP,XMID,YBTM)
    3 CALL LYNE(XMID,YBTM,XLFT,YTOP)
      IF(XSHFT.NE.0) GO TO 4
      XSHFT=.724
      GO TO 5
    4 CONTINUE
      CALL OPTION(1,0,3,0,0)
      CALL CSTRING(.3945,.6381,12HPLEASE PUT$.)
      CALL CSTRING(.3594,.5791,15HTHE FOLLOWING$.)
      CALL CSTRING(.3594,.5264,15HFOUR PLOTS ON$.)
      CALL CSTRING(.418,.4736,10HTHE SAME$.)
      CALL CSTRING(.3828,.4209,13H8 X 10 INCH$.)
      CALL CSTRING(.4531,.3682,7HPRINT$.)
      CALL FRAME
      RETURN
      END


      SUBROUTINE LABPLT(A1,A2,SLEV,IFNCRM)
      CALL MAP(0.,1.,0.,1.,0.,1.,0.,1.)
      CALL OPTION (1,0,3,0,0)
      CALL CSTRING(.383,.6,6HA1 =$.)
      CALL CSTRING(.383,.5,6HA2 =$.)
      CALL CSTRING(.453,.35,3H=$.)
      CALL CNUMBR(.5,.6,A1,4HF8.4)
      CALL CNUMBR(.5,.5,A2,4HF8.4)
      CALL CNUMBR(.5,.35,SLEV,4HF8.4)
      CALL CSTRING(.324,.373,6HSURG$.)
      CALL CSTRING(.312,.326,7HLEVEL$.)
      IF(IFNCRM.EQ.1) GO TO 1
      CALL CSTRING(.383,.747,13HEXPONENTIAL$.)
      GO TO 2
1     CALL CSTRING(.441,.747,8HNORMAL$.)
2     CALL CSTRING(.43,.7,9HPROCESS$.)
      CALL FRAME
      RETURN
      END
```

```
      SUBROUTINE CMAIN (A1,A2,SLEV,DMAR,SURG,OUTPUT,IFNORM)
      DIMENSION DMAR(100),SURG(20,250),OUTPUT(5000)
      INDOUT=0
      ITCTL=5000
      IRUN=ITCTL/20
      RUN=IRUN
      R=.95
      BETA=5.
      ALPHA=0.
      ISRG=0
      ISFST=0
      I1RUN=1
      IPOCH=1
      ICNT=0
      IRUNC=0
      RTCNT=0.
      RTAU=0.
      OLST=0.
      C1=1.
      DO 2 I=1,100
      DMAR(I)=0.
      DO 2 J=1,20
2     SURG(J,I)=0.
3     CONTINUE
      CALL PROCESS (C1,ALPHA,BETA,OUT,IFNORM,A1,A2)
      RTCNT=RTCNT+1.
      RTAU=RTAU+(OLST-OUT)*(OLST-OUT)
      OLST=OUT
      INDOUT=INDOUT+1
      IF(INDOUT.GT.5000) GO TO 11
      OUTPUT(INDOUT)=OUT
      GO TO 12
11    INDOUT=5000
12    IF(IFNORM.EQ.0) GO TO 13
      IOUT=OUT*5.+50.5
      IF(IOUT.LE.1) IOUT=1
      GO TO 14
13    IOUT=(OUT*2.)+1.
14    IF(IOUT.GE.100) IOUT=100
      DMAR(IOUT)=DMAR(IOUT)+1.
      IF(OUT.GE.SLEV) ISRG=1
      IF(ISRG.EQ.1.AND.I1RUN.EQ.1) ISFST=1
      I1RUN=0
      IF(OUT.LT.SLEV.AND.ISRG.EQ.1) GO TO 4
      IF(ISRG.EQ.0) GO TO 3
      ICNT=ICNT+1
      GO TO 3
4     IF(ISFST.EQ.1) GO TO 17
      IF(ICNT.GE.100) ICNT=100
      SURG(IPOCH,ICNT)=SURG(IPOCH,ICNT)+1.
      IRUNC=IRUNC+1
17    ISFST=0
      ISRG=0
      ICNT=0
      GO TO 3
5     IPOCH=IPOCH+1
      IF(IPOCH.GE.21) GO TO 18
      DO 19 I=1,100
      CALL PROCESS (C1,ALPHA,BETA,OUT,IFNORM,A1,A2)
      RTCNT=RTCNT+1.
      RTAU=RTAU+(OLST-OUT)*(OLST-OUT)
      OLST=OUT
      INDOUT=INDOUT+1
      IF (INDOUT.GT.5000) GO TO 20
```

```
      OUTPUT(INDOUT)=OUT
      GO TO 29
20    INDOUT=5000
29    IF(IFNORM.EQ.0) GO TO 27
      IOUT=OUT*5.+50.5
      IF(IOUT.LE.1) IOUT=1
      GO TO 28
27    IOUT=(OUT*2.)+1.
28    IF(IOUT.GE.100) IOUT=100
      DMAR(IOUT)=DMAR(IOUT)+1.
19    CONTINUE
      IIRUN=1
      IRUNC=0
      GO TO 3
18    DO 21 I=1,100
      SUM=0.
      DO 22 J=1,20
22    SUM=SUM+SURG(J,I)
      XBAR=SUM/20.
      IF(XBAR.EQ.0.) GO TO 25
      CI=0.
      DO 23 J=1,20
23    CI=(SURG(J,I)-XBAR)*(SURG(J,I)-XBAR)
      CI=CI/19.
      CI=.4644*SQRT(CI)
      GO TO 26
25    SURG(2,I)=0.
      GO TO 21
26    SURG(2,I)=CI/RUN
21    SURG(1,I)=XBAR/RUN
      WRITE(6,8) RTCNT
8     FORMAT(*1    THE MARGINAL DISTRIBUTION OF THE PROCESS BASED ON *,F8
     1.0,* SAMPLES*)
      WRITE(6,15) A1,A2
15    FORMAT(*0A1=*,F8.4,*     A2=*,F8.4)
      WRITE(6,16)
16    FORMAT(*0*)
      DO 6 I=1,100
      IF(IFNORM.EQ.1) DMAR(I)=5.*DMAR(I)/RTCNT
      IF(IFNORM.EQ.0) DMAR(I)=2.*DMAR(I)/RTCNT
6     WRITE(6,7) I,DMAR(I)
7     FORMAT(I5,F20.8)
      RTAU=RTAU/RTCNT
      SURG(3,1)=ITOTL
      SURG(3,2)=RTCNT
      SURG(3,3)=SLEV
      SURG(3,4)=A1
      SURG(3,5)=A2
      SURG(3,6)=INDOUT
      SURG(3,7)=IFNORM
      CALL OUTPLOT(DMAR,SURG,OUTPUT,IFNORM,RTAU)
      RETURN
      END
```

```
      SUBROUTINE CUTPLOT (DMAR,SURG,OUTPUT,IFNORM,RTAU)
      DIMENSION DMAR(100),SURG(20,250),OUTPUT(5000)
      CALL SETLINE(1)
      CALL INDISP
      BIG=0.
      DO 54 I=1,100
54    IF(DMAR(I).GE.BIG) BIG=DMAR(I)
      YMAX=5.*BIG/4.
      II=100
55    IF(DMAR(II).GT.0.) GO TO 56
      II=II-1
      GO TO 55
56    RII=II
      IBIG=II
      XMAX=RII/2.
      IF(IFNORM.EQ.1) XMAX=RII/5.-9.9
      XMIN=0.
      ISML=1
      IF(IFNORM.EQ.0) GO TO 60
      II=1
62    IF(DMAR(II).GT.0.) GO TO 61
      II=II+1
      GO TO 62
61    RII=II
      ISML=II
      XMIN=(RII-1.)/5.-9.9
60    CALL OPTION(1,0,2,0,0)
      I=SURG(3,2)
      CALL CSTRING(242,973,41HMARGINAL DISTRIBUTION OF          SAMPLES$.)
      CALL CNUMBR (650,973,I,2H16)
      CALL CSTRING(500,75,3HX$.)
      CALL MAPS(XMIN,XMAX,0.,YMAX,.15,.95,.15,.9)
      IF(IFNORM.EQ.0)  GO TO 63
      YTOP=DMAR(1)
      CALL LYNE(XMIN,YTOP,XMIN,0.)
63    DO 57 I=ISML,IBIG
      RI=I
      XSTART=(RI-1.)/2.
      XEND=RI/2.
      IF(IFNORM.EQ.1) XSTART=(RI-1.)/5.-9.9
      IF(IFNORM.EQ.1) XEND=XSTART+.2
      YTOP=DMAR(I)
      IF(I.EQ.100) GO TO 59
      YNT=DMAR(I+1)
      IF(YTOP.GE.YNT) GO TO 59
      CALL LYNE(XEND,YTOP,XEND,YNT)
59    CALL LYNE(XSTART,YTOP,XEND,YTOP)
57    CALL LYNE(XEND,YTOP,XEND,0.)
      CALL OPTION(1,0,2,1,0)
      CALL CSTRING(75,400,14HDENSITY OF X$.)
      CALL FRAME
      INDOUT=SURG(3,6)
      A1=SURG(3,4)
      A2=SURG(3,5)
      CALL SPTM(OUTPUT,INDOUT,A1,A2)
      CALL INDISP
      BIG=0.
      DO 50 I=1,100
50    IF(SURG(1,I).GE.BIG) BIG=SURG(1,I)
      YMAX=5.*BIG/4.
      IF(IFNORM.EQ.0) GO TO 64
      X=SURG(3,3)/1.414213562
      CALL ERFNCTN(X,Y)
      PHI=.5+Y/2.
```

```
      FCTR=(RTAU/(6.2831853*(1-PHI)))*EXP(-.5*SURG(3,3)*SURG(3,3))
      GO TO 66
   64 FCTR=SQRT(RTAU/8.)
   66 CORT=EXP(FCTR/2.)
      YTOP=FCTR*EXP(-1.*FCTR*.5)*CORT
      IF(YTOP.GE.BIG) YMAX=5.*YTOP/4.
      II=100
   51 IF(SURG(1,II).GT.0.) GO TO 52
      II=II-1
      GO TO 51
   52 RII=II
      XMAX=RII+.5
      I=SURG(3,1)
      CALL OPTION(1,0,2,0,0)
      CALL CSTRING(288,973,41HNORMALIZED DISTRIBUTION OF        SURGES$.)
      CALL CNUMBR(707,973,I,2HI5)
      CALL CSTRING(456,75,14HSURGE LEGNTHS$.)
      CALL MAPS(.5,XMAX,0.,YMAX,.15,.95,.15,.9)
      YTOP=SURG(1,1)
      CALL LYNE(.5,0.,.5,YTOP)
      DO 53 I=1,II
      RI=I
      XSTART=RI-.5
      XEND=RI+.5
      YTOP=SURG(1,I)
      IF(I.EQ.100) GO TO 58
      YNT=SURG(1,I+1)
      IF(YTOP.GE.YNT)GO TO 58
      CALL LYNE(XEND,YTOP,XEND,YNT)
   58 CALL LYNE(XSTART,YTOP,XEND,YTOP)
      CALL LYNE(XEND,YTOP,XEND,0.)
      XSTART=XSTART+1./3.
      XEND=XSTART+1./3.
      YTOP=SURG(1,I)+SURG(2,I)
      CALL LYNE(XSTART,YTOP,XEND,YTOP)
      YTOP=SURG(1,I)-SURG(2,I)
   53 CALL LYNE(XSTART,YTOP,XEND,YTOP)
      XINC=RII/500.
      XSTART=.5
      YTOP=CORT*FCTR*EXP(-1.*FCTR*.5)
      DO 65 I=1,500
      RI=I
      XEND=.5+RI*XINC
      YNT=FCTR*CORT*EXP(-1.*FCTR*XEND)
      CALL LYNE(XSTART,YTOP,XEND,YNT)
      XSTART=XEND
   65 YTOP=YNT
      CALL OPTION (1,0,2,1,0)
      CALL CSTRING(75,420,18HNUMBER OF SURGES$.)
      CALL FRAME
      WRITE(6,9)
    9 FORMAT(*1   THE SURGE DISTRIBUTION OF THE PROCESS*)
      WRITE(6,15) SURG(3,4),SURG(3,5)
   15 FORMAT(*0A1=*,F8.4,*    A2=*,F8.4)
      WRITE(6,16)
   16 FORMAT(*0*)
      DO 10 I=1,II
      RI=I
      T1=FCTR*EXP(-1.*FCTR*RI)
      T2=T1*CORT
   10 WRITE(6,11) I,SURG(1,I),SURG(2,I),T1,T2
   11 FORMAT(I5,F15.8,*   +OR-*,F15.8,*   T1=*,F15.8,*   T2=*,F15.8)
      SURG(4,1)=FCTR
      SURG(4,2)=CORT
      CALL CUM (SURG,II,RTAU)
      RETURN
      END
```

```
      SUBROUTINE CUM (SURG,II,RTAU)
      DIMENSION SURG(20,250)
      XMAX=II+1
      CALL INCISP
      CALL OPTION(1,0,2,0,0)
      CALL CSTRING(456,75,14HSURGE LEGNTH$.)
      I=SURG(3,1)
      CALL CNUMBR(707,973,I,2HI5)
      CALL CSTRING(288,973,41HCUMULATIVE DISTRIBUTION OF        SURGES$.)
      CALL OPTION(1,0,2,1,0)
      CALL CSTRING(75,420,18HNUMBER OF SURGES$.)
      CALL MAPS(0.,XMAX,0.,1.2,.15,.95,.15,.9)
      Y=0.
      DO 1 I=1,II
      X=I
      Y=Y+SURG(1,I)
    1 CALL XSTRING(X,Y,3H*$.)
      FCTR=SURG(4,1)
      CORT=SURG(4,2)
      WRITE(6,6) RTAU
    6 FORMAT(*1 THE VALUE OF -R DOUBLE PRIME (0) =*,F15.8)
      WRITE(6,9) FCTR
    9 FORMAT(*0 THE VALUE OF ALPHA =*,F15.8)
      WRITE(6,10) CORT
   10 FORMAT(*0 THE VALUE OF THE INTEGRAL TO ONE HALF =*,F15.8)
      RII=II
      XINC=(RII-.5)/500.
      XSTART=.5
      YTOP=0.
      DO 2 I=1,500
      RI=I
      XEND=RI*XINC+.5
      YNT=1.-CORT*EXP(-1.*FCTR*XEND)
      CALL LYNE(XSTART,YTOP,XEND,YNT)
      XSTART=XEND
    2 YTOP=YNT
      CALL FRAME
      A1=SURG(3,4)
      A2=SURG(3,5)
      IFNORM=SURG(3,7)
      SLEV=SURG(3,3)
      CALL ARROWU(A1,A2,SLEV,IFNORM)
      CALL INCISP
      CALL OPTION(1,0,2,0,0)
      CALL CSTRING(456,75,14HSURGE LEGNTH$.)
      I=SURG(3,1)
      CALL CNUMBR(707,973,I,2HI5)
      CALL CSTRING(288,973,41HCUMULATIVE DISTRIBUTION OF        SURGES$.)
      CALL OPTION(1,0,2,1,0)
      CALL CSTRING(75,420,18HNUMBER OF SURGES$.)
      CALL MAP(0.,1.,0.,1.2,.15,.95,.15,.9)
      CALL LYNE(0.,0.,0.,1.2)
      CALL LYNE(0.,0.,1.,0.)
      CALL LYNE(0.,0.,1.,1.)
      CALL OPTION(1,0,0,0,0)
      DO 3 I=1,13
      RI=I-1
      RI=RI/10.
      CALL LYNE(-.003,RI,.003,RI)
    3 CALL XNUMBR(-.030,RI,RI,4HF3.1)
      XLST=0.
      DO 4 I=1,II
      RI=I
      X=1.-CORT*EXP(-1.*FCTR*RI)
```

```
      DIF=X-XLST
      IF(X.GE..98) GO TO 7
      IF(DIF.LT..03) GO TO 4
      XLST=X
      CALL LYNE(X,-.003,X,.003)
      XST=X
      IF(I.GT.9) XST=X-.005
      CALL XNUMBR(XST,-.022,I,2HI2)
    4 CONTINUE
      CALL OPTION(1,0,2,0,0)
    7 Y=0.
      DO 5 I=1,II
      RI=I
      Y=Y+SURG(1,I)
      X=1.-CORT*EXP(-1.*FCTR*RI)
    5 CALL XSTRING (X,Y,3H*$.)
      CALL FRAME
      RETURN
      END


      SUBROUTINE PROCESS (C1,ALPHA,BETA,CUT,IFNORM,A1,A2)
      IF (C1.EQ.0.) GO TO 1
      V1=A2*(A1*A1-A2*A2-1.)
      V2=A1*A1+A2*A2+1.
      IF(V1.EQ.0.) GO TO 14
      P=V2/V1
      Q=A2/V1
      R=-1./V1
      VA=(3.*Q-P*P)/3.
      VB=(2.*P*P*P-9.*P*Q+27.*R)/27.
      RAD=VB*VB/4.+VA*VA*VA/27.
      IF(RAD.LT.0.) GO TO 5
      IF(RAD.EQ.0.) GO TO 6
      ARG=-VB/2.+SQRT(RAD)
      IF(ARG.LT.0.) GO TO 7
      ACAP=ARG**(1./3.)
      GO TO 8
    7 ACAP=-1.*(-1.*ARG)**(1./3.)
    8 ARG=-VB/2.-SQRT(RAD)
      IF(ARG.LT.0.) GO TO 9
      BCAP=ARG**(1./3.)
      GO TO 10
    9 BCAP=-1.*(-1.*ARG)**(1./3.)
   10 FACT=ACAP+BCAP
      GO TO 11
    6 ARG=-VB/2.
      IF(ARG.LT.0.) GO TO 12
      ACAP=ARG**(1./3.)
      GO TO 13
   12 ACAP=-1.*(-1.*ARG)**(1./3.)
   13 FACT=-ACAP
      GO TO 11
    5 XIM=SQRT(-1.*RAD)
      PKB=-VB/2.
      RMAG=SQRT(PKB*PKB+XIM*XIM)
      ANGL=ATAN2(XIM,PKB)
      REL=RMAG**(1./3.)*COS(ANGL/3.)
      XIMAJ=RMAG**(1./3.)*SIN(ANGL/3.)
      FACT1=2.*REL
      FACT2=-FACT1/2.-XIMAJ*1.732050808-P/3.
      FACT3=-FACT1/2.+XIMAJ*1.732050808-P/3.
      FACT1=FACT1-P/3.
      FACT=1.E50
      IF(FACT1.GT.0..AND.FACT1.LT.FACT) FACT=FACT1
      IF(FACT2.GT.0..AND.FACT2.LT.FACT) FACT=FACT2
      IF(FACT3.GT.0..AND.FACT3.LT.FACT) FACT=FACT3
```

```
      GO TO 15
   11 FACT=FACT-P/3.
      GO TO 15
   14 FACT=1./SQRT(1.+A1*A1+A2*A2)
   15 R1=FACT*A1/(1.-FACT*A2)
      IF(FACT.GT.1.)  WRITE(6,2) A1,A2,FACT
    2 FORMAT(*0  FOR A1 = *,F9.6,*  AND A2 =*,F9.6,* THE SMALLEST POSITI
     1VE ROOT WAS *,E15.8)
      C22=SQRT(1.-R1*R1)
      C1=0.
      NCH=1
      CALL NORM(XNORM,NCH)
      XM2=XNORM
      CALL NORM(XNORM,NCH)
      XM1=R1*XM2+C22*XNORM
    1 CALL NORM(XNORM,NCH)
      X=(A1*XM1+A2*XM2+XNORM)*FACT
      XM2=XM1
      XM1=X
      IF(IFNORM.EQ.0) GO TO 3
      OUT=X
      GO TO 4
    3 X=X/1.414213562
      CALL ERFNCTN(X,Y)
      OUT=BETA*(-1.*ALOG(.5-.5*Y)+ALPHA)
    4 CONTINUE
      RETURN
      END


      SUBROUTINE NORM (XNORM,NCH)
      PI=3.141592654
      IF (NCH.EQ.0) GO TO 1
      U1=RANF(0)
      U2=RANF(0)
      XRO=(SQRT(-2.*ALOG(U1)))*COS(2.*PI*U2)
      XRE=(SQRT(-2.*ALOG(U1)))*SIN(2.*PI*U2)
      XNORM=XRO
      NCH=0
      GO TO 2
    1 XNORM=XRE
      NCH=1
    2 CONTINUE
      RETURN
      END


      SUBROUTINE ERFNCTN (X,Y)
      P=.3275911
      A1=.254829592
      A2=-.284496736
      A3=1.421413741
      A4=-1.453152027
      A5=1.061405429
      SINE=1.
      IF (X.GE.0.) GO TO 1
      SINE=-1.
      X=-1.*X
    1 T=1./(1.+P*X)
      POLY=A1*T+A2*T*T+A3*T*T*T+A4*T**4.+A5*T**5.
      Y=SINE*(1.-POLY*EXP(-X*X))
      X=X*SINE
      RETURN
      END
```

```
      SUBROUTINE SPTM (CUTPUT,INDOUT,A1,A2)
      DIMENSICN CUTPUT(5000)
      SUM=0.
      WRITE(6,4) INDCUT
    4 FORMAT(*1     THE CORRELATION FUNCTICN OF THE PROCESS BASED ON *,I5
     1,* SAMPLES*)
      WRITE(6,8) A1,A2
    8 FORMAT(*0A1=*,F8.4,*        A2=*,F8.4)
      WRITE(6,6)
    6 FORMAT(*0LAGS       CORRELATION*)
      RNDOUT=INDCUT-1
      CALL INCISP
      CALL OPTION(1,0,2,0,0)
      CALL CSTRING(280,973,37HCCRRELATION FUNCTION OF THE PROCESS$.)
      CALL CSTRING(496,75,10HTIME LAGS.)
      CALL MAPS(0.,25.,-1.5,1.5,.15,.95,.15,.9)
      CALL LYNE(0.,0.,25.,0.)
      XOLD=0.
      YOLD=0.
      DO 1 I=1,INDOUT
    1 SUM=SUM+OUTPUT(I)
      RN=INDOUT
      AVE=SUM/RN
      DO 2 I=1,26
      IM1=I-1
      SUM=0.
      ISTOP=INDOUT-IM1
      DO 3 IT=1,ISTOP
      INDX=IT+IM1
    3 SUM=SUM+(OUTPUT(IT)-AVE)*(OUTPUT(INDX)-AVE)
      SPECT=SUM/RN
      IF(IM1.EC.0) SPZERO=SPECT
      SPECT=SPECT/SPZERO
      RX=IM1
      CALL LYNE(XCLD,YOLD,RX,SPECT)
      XOLD=RX
      YOLD=SPECT
    2 WRITE(6,5) IM1,SPECT
    5 FORMAT (I5,F20.8)
      WRITE(6,7) SPZERC
    7 FORMAT(*0  THE VARIANCE OF THIS SAMPLE WAS *,F15.8)
      CALL OPTION (1,0,2,1,0)
      CALL CSTRING(75,360,25HCORRELATION COEFFICIENTS.)
      CALL FRAME
      RETURN
      END
```

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